THE TEACHER'S CIRCLE - June 19, 2007
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FRACTIONS, DECIMALS, RATIOS, RATES, PERCENTS, PROPORTIONS.

Foreword: A Real World Story

A man had an unlimited data plan in the US and called to check on the roaming rates before going to Canada. The rep told him it was point zero zero two cents per kilobyte. He asked for clarification that it was really 0.002 cents/KB and not 0.002 dollars/KB and the rep said yes, it is 0.002 cents/KB. Then he got the rep to write in the file that he was being quoted 0.002 cents/KB. He went to Canada, used 35000KB of traffic, and went back home to a bill of $70, not 70 cents. So he called Verizon customer service, tried to explain to the rep that he had been misquoted, and the rep couldn't get it. So he got bumped to a supervisor, and started recording the call.

The following is a part of the real dialog with the supervisor.

The Customer (C): Do you recognize that there is a difference between one dollar and one cent?

Verizon Floor Manager (V): Definitely.

C: Do you recognize that there is a difference between half a dollar and half a cent?

V: Definitely.

C: Then do you therefore recognize that there is a difference between point zero zero two dollars and point zero zero two cents?

V: No

You can see the whole story at http://verizonmath.blogspot.com/ and listen to the entire dialog at http://pdos.lcs.mit.edu/~rsc/verizon-math.mp3

What's between a foreword and an introduction? Obviously a

Problem:

Find a fraction strictly between $\frac{48}{97}$ and $\frac{49}{99}$ with the smallest possible denominator.

Introduction: Some definitions.

A (positive) *fraction* $\frac{m}{n}$ is a point on the number line: we divide every unit line segment into n equal parts, then take m of these parts. It can – and should – be shown that $\frac{m}{n} = m \div n$.

(This relationship between fractions and division is crucial. It should not be skipped or shortchanged. Unless it is thoroughly understood the whole idea of a fraction will be murky and dealing with fractions will remain scary and painful.)

Two fractions are *equal* ($\equiv$*equivalent*) if they are the same point. Clearly,

$$\frac{ac}{bc} = \frac{a}{b}$$

Indeed, if we divide a unit into c times as many parts and take c times as many of them we will arrive at the same point, right?.

Once positive fractions are defined, we can take their negatives as points at the same distance from 0 and on the opposite side of it. Positive and negative fractions constitute *rational* numbers. From now on, I'll use a 'fraction' and a 'rational number' as synonyms.

Since fractions are just points on a number line, we can compare them as usual: $\frac{a}{b} < \frac{c}{d}$ if $\frac{a}{b}$ is to the left of $\frac{c}{d}$. How can that be decided? Just divide units into so many parts (bd will do) that we can take a counting number of these parts to arrive at each of the two fractions.

The following is the *definition* of various arithmetic operations of fractions.

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}; \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

These definitions may be motivated by discussing lengths on the number line and area of a rectangle, respectively.

Decimals are simply fractions with denominators that are powers of 10.
(This is certainly true about finite ($\equiv$terminating) and repeating decimals. Non-repeating decimals represent irrational numbers and this subject is quite subtle. It might be a good idea to mention that there are some points (a lot of them!) on the number line that cannot be reached by the process of dividing unit segments into any number of parts. A good example is, of course, $\sqrt{2}$. But this whole topic can be postponed until students are completely at home with fractions.)

We say that a is c percent of b if $\frac{a}{b} = \frac{c}{100}$.

(*What follows is almost a quote from* [1].

Why should anyone bother learning about percent? It is because society has miraculously reached the consensus of using percent to express part-whole relations in everyday life. Comparing $\frac{5}{7}$ with $\frac{11}{15}$, for example, is unpleasant, but if we write $\frac{5}{7} = 71\frac{3}{7}\%$ and $\frac{11}{15} = 73\frac{1}{3}\%$, it's clear that $\frac{5}{7} < \frac{11}{15}$.

If a and b are fractions, the *ratio* of a and b is simply $\frac{a}{b}$ (and thus just a fraction, too). Two pairs of numbers *are in the same ratio* if $\frac{a}{b} = \frac{c}{d}$.

A *rate* is the ratio of two numbers each referring to a different unit, e.g., a car driving 100 miles in 2 hours drives at a rate of $\frac{100}{2}$ miles per hour.

Part I: My favorite problems.

1. If a driver drives one mile at 30 miles per hour, how fast should he drive another mile so that his average speed on this 2-mile trip is 60 miles per hour?

2. A brick weighs 1 pound and half the brick. How many pounds does the brick weigh?

3. Suppose we have a barrel of wine and a cup of tea. A teaspoon of wine is taken from the barrel and poured into the cup of tea. Then the same teaspoon of the mixture is taken from the cup and poured into the barrel. Now the barrel contains some tea and the cup contains some wine. Which volume is larger – that of the tea in the wine barrel or of the wine in the teacup?

Note: The same question could be asked after the process has been repeated several times.

4. At sunrise two old women started to walk towards each other. One started from point A and went towards point B while the other started at B and went towards A. They met at noon but did not stop; each one continued to walk maintaining her speed and direction. The first woman came to the point B at 4:00 pm, and the other one came to point A at 9:00 pm. At what time did sun rise that day?

5. A car and a van are 180 miles apart on a straight road. The drivers start driving toward each other at noon, the car at 50 mi/h, and the van at 40 mi/h. A fly starts from the front bumper of the van at noon and flies to the bumper of the car, then immediately back to the bumper of the van, back to the car, and so on, until the car and the van meet. If the fly flies at a speed of 100 mi/h, how far does the fly fly?

Part II: Problems from AMC 8.

1. (AJHSME 1986) Which of the following numbers has the largest reciprocal?

(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) 1 (D) 5 (E) 1986

2. (AJHSME 1986) If $200 \le a \le 400$ and $600 \le b \le 1200$, then the largest value of the quotient $\frac{b}{a}$ is

(A) $\frac{3}{2}$ (B) 3 (C) 6 (D) 300 (E) 600

3. (AJHSME 1994) Which of the following is the largest?

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{5}{12}$ (E) $\frac{7}{24}$

4. (AJHSME 1994) The number halfway between $\frac{1}{6}$ and $\frac{1}{4}$ is

(A) $\frac{1}{10}$ (B) $\frac{1}{5}$ (C) $\frac{5}{24}$ (D) $\frac{7}{24}$ (E) $\frac{5}{12}$

5. (AJHSME 1997) Which of the following numbers is larger?

(A) 0.97 (B) 0.979 (C) 0.9709 (D) 0.907 (E) 0.9089

6. (AJHSME 1997) In the number 74982.1035 the value of the *place* occupied by the digit 9 is how many times as great as the value of the *place* occupied by the digit 3?

(A) 1,000 (B) 10,000 (C) 100,000 (D) 1,000,000 (E) 10,000,000

7. (AJHSME 1998) For $x = 7$, which of the following is smallest?

(A) $\frac{6}{x}$ (B) $\frac{6}{x+1}$ (C) $\frac{6}{x-1}$ (D) $\frac{x}{6}$ (E) $\frac{x+1}{6}$

8. (AJHSME 1997) Which of the following numbers is largest?

(A) 9.12344 (B) $9.123\overline{4}$ (C) $9.12\overline{34}$ (D) $9.1\overline{234}$ (E) $9.\overline{1234}$

9. (AMC 8, 1999) The third exit on a highway is located at milepost 40 and the tenth exit is at milepost 160. There is a service center on the highway located three-fourths of the way from the third exit to the tenth exit. At what milepost would you expect to find this service center?

(A) 90 (B) 100 (C) 110 (D) 120 (E) 130

10. (AMC 8, 2000) Which of these numbers is less than its reciprocal?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

11. (AMC 8, 2000) How many whole numbers lie in the interval between $\frac{5}{3}$ and 2π?

(A) 2 (B) 3 (C) 4 (D) 5 (E) infinitely many

12. (AJHSME 1994) $\frac{1}{10}+\frac{2}{10}+\frac{3}{10}+\frac{4}{10}+\frac{5}{10}+\frac{6}{10}+\frac{7}{10}+\frac{8}{10}+\frac{9}{10}+\frac{55}{10}=$

(A) $4\frac{1}{2}$ (B) 6.4 (C) 9 (D) 10 (E) 11

13. (AJHSME 1996) Let x be the number $0.\underbrace{0000...0000}_{1996 \text{ zeros}}1$, where there are 1996 zeros after the decimal point. Which of the following expressions represents the largest number?

(A) $3+x$ (B) $3-x$ (C) $3\cdot x$ (D) $3/x$ (E) $x/3$

14. (AJHSME 1997) $\frac{1}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{7}{10000}=$

(A) 0.0026 (B) 0.0197 (C) 0.1997 (D) 0.26 (E) 1.997

15. (AJHSME 1998) $100\times19.98\times1.998\times1000=$

(A) $(1.998)^2$ (B) $(19.98)^2$ (C) $(199.8)^2$ (D) $(1998)^2$ (E) $(19980)^2$

16. (AJHSME 1986) $\frac{2}{1-\frac{2}{3}}=$

(A) -3 (B) $-\frac{4}{3}$ (C) $\frac{2}{3}$ (D) 2 (E) 6

17. (AJHSME 1995) Which of the following operations has the same effect on a number as multiplying by $\frac{3}{4}$ and then dividing by $\frac{3}{5}$?

(A) dividing by $\frac{4}{3}$ (B) dividing by $\frac{9}{20}$ (C) multiplying by $\frac{9}{20}$

(D) dividing by $\frac{5}{4}$ (E) multiplying by $\frac{5}{4}$

18. (AJHSME 1996) $\frac{2+4+6+...+34}{3+6+9+...+51}=$

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\frac{17}{3}$ (E) $\frac{34}{3}$

19. (AJHSME 1996) The remainder when the product $1492\cdot1776\cdot1812\cdot1996$ is divided by 5 is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

20. (AJHSME 1997) If the product $\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{a}{b} = 9$, what is the sum of a and b?

(A) 11 (B) 13 (C) 17 (D) 35 (E) 37

21. (AJHSME 1998) $\dfrac{\frac{3}{8}+\frac{7}{8}}{\frac{4}{5}} =$

(A) 1 (B) $\frac{25}{16}$ (C) 2 (D) $\frac{43}{20}$ (E) $\frac{47}{16}$

22. (AJHSME 1997) $2\left(1-\frac{1}{2}\right)+3\left(1-\frac{1}{3}\right)+4\left(1-\frac{1}{4}\right)+\ldots+10\left(1-\frac{1}{10}\right)=$

(A) 45 (B) 49 (C) 50 (D) 54 (E) 55

23. (AMC 8, 2000) The operation $\otimes$ is defined for all nonzero numbers by $a \otimes b = \frac{a^2}{b}$. Determine $[(1 \otimes 2) \otimes 3] - [1 \otimes (2 \otimes 3)]$.

(A) $-\frac{2}{3}$ (B) $-\frac{1}{4}$ (C) 0 (D) $\frac{1}{4}$ (E) $\frac{2}{3}$

24. (AJHSME 1990) Which of these five numbers is the largest?

(A) $13579+\frac{1}{2468}$ (B) $13579-\frac{1}{2468}$ (C) $13579\times\frac{1}{2468}$

(D) $13579\div\frac{1}{2468}$ (E) 13579.2468

25. (AJHSME 1994) Let W, X, Y and Z be four different digits selected from the set {1, 2, 3, 4, 5, 6, 7, 8, 9}. If the sum $\frac{W}{X}+\frac{Y}{Z}$ is to be as small as possible, then $\frac{W}{X}+\frac{Y}{Z}$ must equal

(A) $\frac{2}{17}$ (B) $\frac{3}{17}$ (C) $\frac{17}{72}$ (D) $\frac{25}{72}$ (E) $\frac{13}{36}$

26. (AJHSME 1995) Find the smallest whole number that is larger than the sum

$$2\frac{1}{2}+3\frac{1}{3}+4\frac{1}{4}+5\frac{1}{5}.$$

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

27. (AJHSME 1995) At Clover View Junior High, one half of the students go home on the school bus. One fourth go home by automobile. One tenth go home on their bicycles. The rest walk home. What fractional part of the students walk home?
(A) $\frac{1}{16}$ (B) $\frac{3}{20}$ (C) $\frac{1}{3}$ (D) $\frac{17}{20}$ (E) $\frac{9}{10}$

28. (AJHSME 1996) When Walter drove up to the gasoline pump, he noticed that his gasoline tank was 1/8 full. He purchased 7.5 gallons of gasoline for \$10. With this additional gasoline, his gasoline tank was then 5/8 full. The number of gallons of gasoline his tank holds when it is full is
(A) 8.75 (B) 10 (C) 11.5 (D) 15 (E) 22.5

29. (AJHSME 1998) Each of the letters W, X, Y, and Z represents a different integer in the set {1, 2, 3, 4}, but not necessarily in that order. If $\frac{W}{X}-\frac{Y}{Z}=1$, then the sum of W and Y is
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

30. (AMC 8, 1999) The average age of the 40 members of a computer science camp is 17 years. There are 20 girls, 15 boys, and 5 adults. If the average age of the girls is 15 and the average age of the boys is 16, what is the average age of the adults?
(A) 26 (B) 27 (C) 28 (D) 29 (E) 30

31. (AMC 8, 2000) There is a list of seven numbers. The average of the first four numbers is 5, and the average of the last four numbers is 8. If the average of all seven numbers is $6\frac{4}{7}$, then the number common to both sets of four numbers is
(A) $5\frac{3}{7}$ (B) 6 (C) $6\frac{4}{7}$ (D) 7 (E) $7\frac{3}{7}$

32. (AJHSME 1986) Sale prices at the Ajax Outlet Store are 50% below original prices. On Saturdays an additional discount of 20% off the sale price is given. What is the Saturday price of a coat whose original price is \$180?
(A) \$54 (B) \$72 (C) \$90 (D) \$108 (E) \$110

33. (AJHSME 1990) A dress originally priced at \$80 was put on sale at 25% off. If a 10% tax was added to the sale price, then the total selling price of the dress was
(A) \$45 (B) \$52 (C) \$54 (D) \$66 (E) \$68

34. (AJHSME 1994) A shopper buys a \$100 coat on sale for 20% off. An additional \$5 is taken off the sale price by using a discount coupon. A sales tax of 8% is paid on the final selling price. The total amount the shopper pays for the coat is
(A) \$81.00 (B) \$81.40 (C) \$82.00 (D) \$82.08 (E) \$82.40

35. (AJHSME 1995) A jacket and a shirt originally sold for \$80 and \$40, respectively. During a sale Chris bought the \$80 jacket at a 40% discount and the \$40 shirt at a 55% discount. The total amount saved was what percent of the total of the original prices?

(A) 45% (B) $47\frac{1}{2}\%$ (C) 50% (D) $79\frac{1}{6}\%$ (E) 95%

36. (AJHSME 1995) A team won 40 of its first 50 games. How many of the remaining 40 games must this team win so it will have won exactly 70% of its games for the season?
(A) 20 (B) 23 (C) 28 (D) 30 (E) 35

37. (AJHSME 1996) Ana's monthly salary was \$2000 in May. In June she received a 20% raise. In July she received a 20% pay cut. After the two changes in June and July, Ana's monthly salary was
(A) \$1920 (B) \$1980 (C) \$2000 (D) \$2020 (E) \$2040

38. (AJHSME 1997) At the grocery store last week, small boxes of facial tissue were priced at 4 boxes for \$5. This week they are on sale at 5 boxes for \$4. The percent decrease in the price per box during the sale was closest to
(A) 30% (B) 35% (C) 40% (D) 45% (E) 65%

39. (AJHSME 1997) Three bags of jelly beans contain 26, 28, and 30 beans. The ratio of yellow beans to all beans in each of these bags are 50%, 25%, and 20%, respectively. All three bags of candy are dumped into one bowl. Which of the following is closest to the ratio of yellow jelly beans to all beans in the bowl?
(A) 31% (B) 32% (C) 33% (D) 35% (E) 95%

40. (AJHSME 1998) At Annville Junior High School, 30% of the students in the Math Club are in the Science Club, and 80% of the students in the Science Club are in the Math Club. There are 15 students in the Science Club. How many students are in the Math Club?
(A) 12 (B) 15 (C) 30 (D) 36 (E) 40

41. (AMC 8, 2000) Ara and Shea were once the same height. Since then Shea has grown 20% while Ara has grown half as many inches as Shea. Shea is now 60 inches tall. How tall, in inches, is Ara now?
(A) 48 (B) 51 (C) 52 (D) 54 (E) 55

42. (AJHSME 1986) In July 1861, 366 inches of rain fell in Cherrapunji, India. What was the average rainfall in inches per hour during that month?

(A) $\frac{366}{31\times 24}$ (B) $\frac{366\times 31}{24}$ (C) $\frac{366\times 24}{31}$ (D) $\frac{31\times 24}{366}$ (E) $366\times 31\times 24$

43. (AJHSME 1986) At the beginning of a trip, the mileage odometer read 56,200 miles. The driver filled the gas tank with 6 gallons of gasoline. During the trip, the driver filled his tank again with 12 gallons of gasoline when the odometer read 56,560. At the end of the trip, the

driver filled the tank again with 20 gallons of gasoline. The odometer read 57,060. To the nearest tenth, what was the car's average miles-per-gallon for the entire trip?
(A) 22.5 (B) 22.6 (C) 24.0 (D) 26.9 (E) 27.

44. (AJHSME 1994) Two children at a time can play pairball. For 90 minutes, with only two children playing at one time, five children take turns so that each one plays the same amount of time. The number of minutes each child plays is
(A) 9 (B) 10 (C) 18 (D) 20 (E) 36

45. (AJHSME 1995) Students from three middle schools worked on a summer project. Seven students from Allen School worked for 3 days. Four students from Balboa School worked for 5 days. Five students from Carver School worked for 9 days. The total amount paid for the students' work was $774. Assuming each student received the same amount for a day's work, how much did the students from Balboa School earn altogether?
(A) $9.00 (B) $48.38 (C) $180.00 (D) $193.50 (E) $258.00

46. (AJHSME 1997) A two-inch cube ($2\times2\times2$) of silver weighs 3 pounds and is worth $200. How much is a three-inch cube of silver worth?
(A) $300 (B) $375 (C) $450 (D) $560 (E) $675

47. (AJHSME 1998) A child's wading pool contains 200 gallons of water. If water evaporates at the rate of 0.5 gallons per day and no other water is added or removed, how many gallons of water will be in the pool after 30 days?
(A) 140 (B) 170 (C) 185 (D) 198.5 (E) 199.85

48. (AJHSME 1990) Three Δ's and a $\Diamond$ will balance nine $\bullet$'s. One Δ will balance a $\Diamond$ and a $\bullet$. How many $\bullet$'s will balance the two $\Diamond$'s ?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

49. (AJHSME 1994) Given that 1 mile = 8 furlongs and 1 furlong = 40 rods, the number of rods in one mile is
(A) 5 (B) 320 (C) 660 (D) 1760 (E) 5280

50. (AJHSME 1995) An American traveling in Italy wishes to exchange American money (dollars) for Italian money (lire). If 3000 lire = $1.60, how many lire will the traveler receive in exchange for $1.00?
(A) 180 (B) 480 (C) 1800 (D) 1875 (E) 4875

51. (AMC 8, 1999) In a far-off land three fish can be traded for two loaves of bread and a loaf of bread can be traded for four bags of rice. How many bags of rice is one fish worth?
(A) $\frac{3}{8}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $2\frac{2}{3}$ (E) $3\frac{1}{3}$

Part III: Good sources of problems and strategies.

1. The Mathematics Pre-Service Teachers Need to Know, R. James Milgram,
   the entire book is at http://math.stanford.edu/ftp/milgram/FIE-book.pdf

NOTE: This is a 564-page book that may take several minutes to download. When the download is complete, the full book can be viewed in Adobe Acrobat Reader.

2. First Steps for Math Olympians: Using the American Mathematics Competitions, J. Douglas Faires, MAA Problem Books, 2006

NOTE: Problems used in this book are taken from AHSME, AMC 10, and AMC 12.

Afterword: Problems Going in Circles.

After the last summer's Teacher's Circle workshop, the participants stayed in touch. What follows is a part of an on-going discussion.

1. George's question:

Hi all,

I am transitioning from operations with fractions and mixed numbers into ratios and proportions. I have those few students that struggle with fractions. They get 3/8 + 1/8 =4/8 or 1/2 but then they say that 1/2 + 1/3 + 1/6 equals 3/11. Nevertheless, 95% of my students understand a common denominator is needed before + or -.

What challenge problems of interest would you have for ratios or proportions? Is 6th grade to early for the golden ratio? I guess it depends on what I try and do with it. Any help is welcomed.

2. Ann's reply:

Here's a problem on proportional reasoning that I used with sixth graders. I found it somewhere and revised the numbers so that the arithmetic was easier, hoping the kids would focus on the reasoning issue. I posed it as a warm-up as we started the unit on proportions, thinking that the kids could work on it for a few minutes and then we could discuss it, and that I would get some insight into their understanding of proportional reasoning. That was a mistake-- we ended up spending half a class period working on it, then I asked them to write about it for homework. We spent a lot of time discussing it the next day, too. Now I might pose the problem at the beginning of the unit but not ask them to solve it until the end, or use it as a POW.

Here's the problem:

Amy and Katie went shopping for shoes at Pay Less Shoe Store, which was having a sale. The sign in the window said: "Buy one pair, get the second pair (of equal or lesser value) for half price. Discount taken at the register." Amy found a pair of shoes she liked marked $45, and Katie found a pair she liked marked $30. When they got to the register, the cashier rang up the sale and said "That will be $60." How much of the $60 should each girl pay?

My students suggested that each girl should pay half, that they should split the $15 discount equally, that Amy should pay full price because she picked the more expensive pair of shoes and Katie should get the discount, and that one should pay for both pairs of shoes and the other should buy lunch. No one agreed with me that Amy should pay $36 and Katie should pay $24, but we had an interesting discussion.

3. Sam's comment:

I liked Ann's idea very much. Here is another thought which would complement that one, since it is more of a puzzle than a real application. Challenge students to use the numbers 96, 97, 98, and 99 to create two fractions with the smallest possible sum. (For example, one way to arrange the numbers would be 99/97+96/98.) This is a nasty computation, of course, so lead them to formulate and try out smaller sets of consecutive positive integers. In other words, guide them to try out the same thing with 1, 2, 3, 4 or 2, 3, 4, 5 or other examples involving small numbers until they observe a pattern. They can then use their pattern to predict that the smallest sum will be 97/99+96/98. (Which is perhaps not what one might expect!)

Incidentally, this also makes a motivation for learning how to manipulate fractions involving variables, because to prove the conjecture students need to compare the three quantities n/(n+3) + (n+1)/(n+2), n/(n+2) + (n+1)/(n+3), and n/(n+1) + (n+2)/(n+3).

4. Ned's reaction:

THANKS, SAM!!! That is a great problem.

Question: I wrote up this little worksheet (see below) that summarizes what you wrote. Do you think I am leading 7th graders too much by writing these questions. I am curious what people think.

Worksheet Piggy-backing off of Sam…
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1 Use each of the numbers 96, 97, 98, and 99 (without your calculator) to create the sum of two fractions. How many different sums of two fractions can you make? Show them all without computing the sum. For example, one way to arrange the numbers would be 99/97 + 96/98.

2 Now that you have written out all of the possible sums of two fractions, which sums can be eliminated with just a quick glance of the eye as not being the smallest? What is your reasoning?

3 Find the scenario that makes the smallest sum and compute it. You might, of course, want to try out smaller sets of consecutive positive integers (like 1, 2, 3, 4 or 2, 3, 4, 5) until you observe a pattern and then apply your reasoning to 96, 97, 98, 99.

4 Now, what if your numbers were n, n+1, n+2, n+3 and you had to create two fractions with the smallest possible sum. What would those two fractions be? Prove that your answer is true.

5. Sam's reply:

Hi Ned,

I'm glad you liked the problem. I think you've developed very nicely into a worksheet as well! Here are a few thoughts I had as I read over it:
** Good idea having them write down all possible sums, and eliminate some without further computation.
** I would assume that students are not going to think of trying smaller examples and just instruct them directly to solve the question for 1, 2, 3, 4 and 2, 3, 4, 5 then look for patterns so that they can predict the answer for n, n+1, n+2, n+3.
** There is a quantum leap from these parts to the one in which they prove their answer is correct. I'm not sure what to do about that.. maybe leave it off for now, or come up with a different way to conclude the worksheet?

I like your worksheet, and hope it's useful for you!

6. Harold's input:

4. **Fabulous Fractions**

(a) Find four different decimal digits a, b, c, d so that $\frac{a}{b} + \frac{c}{d} < 1$ and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

(b) Thanks to Sam Vandervelde for this problem. Use exactly eight digits to form four two digit numbers $\underline{ab}, \underline{cd}, \underline{ed}, \underline{gh}$ so that the sum $\frac{\underline{ab}}{\underline{cd}} + \frac{\underline{ef}}{\underline{gh}}$ is as small as possible. As usual, interpret $\underline{ab}$ as $10a + b$, etc.

(c) Use exactly eight nonzero digits to form four two digit numbers $\underline{ab}, \underline{cd}, \underline{ed}, \underline{gh}$ so that the sum $\frac{\underline{ab}}{\underline{cd}} + \frac{\underline{ef}}{\underline{gh}}$ is as small as possible. As usual, interpret $\underline{ab}$ as $10a + b$, etc.

(d) Next find six different decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} = \frac{e}{f}$.

(e) Next find six different decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1$ and the sum is as large as possible.

Here is another little conversation.

1. Beth's question:

I am writing to take advantage of your offer to create a math problem for my class. I teach 6th grade and we are studying decimals and percents.
Thanks a million!

2. Josh's reply:

I'm not sure what level your students are at, but the connection between decimals, percents, and fractions is certainly VERY rich.

I think for percents: list ALL the percents 1 through 100 on a piece of paper (you provide this for them) and then next to that you have a space for the REDUCED fraction. Then you ask which denominators appear in that list, and why? (you get all the divisors of 100, of course!). Then ask what would happen if we had "per mille" instead of "per cent" -- what denominators would you get then?

Do they do a lot with calculators? That might lead to some other interesting explorations. For example, you might lead them to the golden ratio by asking something like "20% of 120 is less than 100, so 20 is too small. 80% of 180 is more than 100, so 80 is too big. 50% of 150 is less than 100, so 50 is too small. Is there a number that's just right?" and the answer is about 61.8% of 161.8 being very very close to 100 (but of course the real answer is irrational so they can't state it exactly in this form).

You can also, going into repeating decimals, look into how long it takes for the decimal to repeat. There are some great patterns relating the number of digits in the repeating part and the denominator of the fraction. That's a fun exploration if you're interested in that aspect of decimals and percents.

There's also the lead-in to calculus ... "Mr. Smith is working on painting his house. He does 40% of the work every day. How many days does it take to paint the whole room?" With some work, the students eventually figure out that it means 2 1/2 days. But then you explain, no, 40% on the first day does 40% of the house, but 40% on the second day only does 40% OF the 60% that's left unpainted ... then you can get into some interesting patterns, you can draw pictures in a rectangle, and stuff like that. (You can do 50% instead of 40% if you want to keep things a little simpler).

I hope someone else in the teachers circle group can add a few more ideas. I have a couple other resources i can look at for some more ideas, too. Let me know what you think of these!

3. Tom's comment:

I like Josh's suggestions.

I actually got hooked on the "conversion of fractions to decimals" problem and have used that in a couple of math circles. I have a handout here: http://www.geometer.org/mathcircles/fractions.pdf

Some of the material there may be too advanced, but some is not, and some can probably be converted to a form that's usable.

One interesting thing to do that may get the kids to learn something is to have them use calculators to do "long division" to find out, for example, just how long it takes for 1/31 to repeat. Most calculators won't show enough decimals and you have to figure out how to get more.

If you do the division 1/31, say the calculator shows: .032258xxx, where the xxx part is a few more numbers.

Multiply just .032258 by 31 and you'll get a number close to 1, but the error will just be what would be left over if you'd done long division to 6 places. You can then divide 31 into that error to get the next 6 places and so on. It's like doing long division 6 digits at a time. Obviously it would be better to learn to do this on something where the answer is known, like:

1/7 = .142857142857142857...

Use a calculator to get 4 digits at a time to see how it works.