Oakland/East Bay Teacher's Circle
Magic, Latin, and Sudoku Squares

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4	9	2
3	5	7
8	1	6

12	1	14	7
6	15	4	9
3	10	5	16
13	8	11	2

A *traditional magic square* is an $n \times n$ square containing the numbers $1, 2, \ldots, n^2$, such that every row, column, and main diagonal sums up to the same number, the *magic sum*. Here are some warm-up problems:

(1) Are there any magic 2×2 squares? Explain.

(2) How many magic 3×3 squares are there?

(3) Find a formula for the magic sum of an $n \times n$ square.

Next we'll do some counting. Let's relax the conditions on a square to be magic...

We define a *(weak) semimagic square* to be a square matrix whose entries are nonnegative integers and whose rows and columns (called *lines* in this setting) sum to the same number. A *(weak) magic square* is a semimagic square whose two main diagonals also add up to the line sum.

3	0	0
0	1	2
0	2	1

1	2	0
0	1	2
2	0	1

Our goal is to count these squares. In the traditional case (where we're only allowed to use the numbers 1 through n^2, each exactly once), this is in some sense boring:[1] for each order there is a fixed number of traditional magic squares; for example, there are 7040 traditional 4×4 magic squares. The situation becomes more interesting if we weaken our conditions and study the number of magic squares as a function of the line sum. Here are a few problems.

[1] It is, nevertheless, an incredibly hard problem to count all traditional magic squares of a given size n. At present, these numbers are only known for $n \leq 5$—see Neil Sloane's *Integer Sequence Encyclopedia* entry `http://www.research.att.com/projects/OEIS?Anum=A006052`.

(1) How many 1×1 semimagic squares are there with line sum t?

(2) How many 1×1 magic squares are there with line sum t?

(3) How many 2×2 semimagic squares are there with line sum t?

(4) How many 2×2 magic squares are there with line sum t?

(5) Find a formula for the number of semimagic $n \times n$ squares with line sum 1.

(6) Let $(x_{ij})_{1 \leq i,j \leq 3}$ be a magic 3×3 square.

   (a) Show that the center term x_{22} is the average over all x_{ij}.

   (b) Show that there are no magic 3×3 squares with line sum t, if t is not a multiple of 3.

(7) The next two exercises are only fun if you know some linear algebra or, alternatively, have some feel for dimensions. Let's denote the total number of semimagic and magic squares of order n and line sum t by $S_n(t)$ and $M_n(t)$, respectively. Prove that $S_n(t)$ grows like a polynomial of degree $(n-1)^2$ or, alternatively, that the dimension of

$$\left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : \ x_{jk} \geq 0, \ \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\} \tag{1}$$

equals $(n-1)^2$. (You may start with the cases $n = 1, 2, 3$ to get a feel for this dimension computation.)

(8) Prove that $M_n(t)$ grows like a polynomial of degree $(n-1)^2 - 2$.

A few magic remarks... The problem of counting magic squares (other than traditional magic squares) seems to have occurred to anyone only in the twentieth century, no doubt because there was no way to approach the question previously. The first formulas—beyond our exercises—addressing the counting problem, namely

$$S_3(t) = \frac{1}{8}\,t^4 + \frac{3}{4}\,t^3 + \frac{15}{8}\,t^2 + \frac{9}{4}\,t + 1$$

and

$$M_3(t) = \begin{cases} \frac{2}{9}\,t^2 + \frac{2}{3}\,t + 1 & \text{if } 3|t, \\ 0 & \text{otherwise,} \end{cases}$$

were established by Percy Macmahon in 1915. You might have noticed that $S_1(t)$, $S_2(t)$, and $S_3(t)$ are all *polynomials* in t. This evidence suggests that $S_n(t)$ is, in fact, a polynomial for all n, a theorem from 1973 which is independently due to Eugène Ehrhart and Richard Stanley. Our examples of $M_2(t)$ and $M_3(t)$ show already that $M_n(t)$ is in general *not* a polynomial. In fact, there is plenty of reason to conjecture that $M_n(t)$ is not a polynomial for *all* $n > 1$, but this conjecture is still open for $n \geq 6$.

The sets of the form (1) are the *Birkhoff polytopes* B_n, named after Garrett Birkhoff, who proved that the extremal points of B_n are the permutation matrices. A long-standing open problem is the determination of the (relative) volume of B_n, which is known only for $n \leq 10$. This volume equals—as you may try to prove—the leading coefficient of $S_n(t)$, a fact that asserts that the computation of $S_n(t)$ is generally hard.

Ok—it's time to play Sudoku...

			6					5
				5		6		
	5			8				9
	2			1	6	7		
		1	7			9	6	
7	6			2				3
	9		3				2	7
	3	7		4				6
6			9		1	3		

	1							9
	5						7	
		7		3	8			1
				5		4		
5	4		1			3		
				4			1	
3	7			9	2		8	6
8	6			1	4	5	3	

A *Sudoku square* is a 9×9 grid filled with nine symbols (such as the numbers from 1 to 9) in such a way that each row, column, and the nine 3×3 subsquares (shown above) contain each symbol exactly once. If we leave out the last condition, we get the definition of a *Latin square*, i.e., Sudoku squares are special cases of Latin squares. There are quite a number of famous long-standing open problems connected to Latin squares.

In a *Sudoku puzzle*, the square contains some entries (called *clues*), and the goal is to complete the Sudoku square. In the classical setting, the clues are chosen such that there is only one way to complete each square. You may try your luck with the two Sudoku puzzles above (caveat: If you've never played a Sudoku puzzle, watch out—these squares are addictive). We will work on two problems regarding Sudoku squares:

(1) How many Sudoku squares are there?

(2) What is the minimum number of clues that yield a unique solution to a Sudoku puzzle?

These are hard questions. In fact, (1) was answered only last year (the number of Sudoku squares is 6 670 903 752 021 072 936 960), and (2) remains open (it is conjectured that the minimum number is 17). So we will simplify the problems and work with 4×4 Sudoku squares. The next page contains many empty 4×4 squares for you to experiment with questions (1) and (2) in the 4×4 case.

A few websites

http://mathworld.wolfram.com/MagicSquare.html (basic definitions)

http://mathforum.org/alejandre/magic.square.html (classroom activities, lots of links)

http://pasles.org/Franklin/index.html (Franklin squares)

http://mathforum.org/te/exchange/hosted/suzuki/MagicSquare.html (many many examples, lots of links)

http://www.grogono.com/magic/index.php (construct your own magic square)

http://www.jcu.edu/math/vignettes/magicsquares.htm (nice description of construction)

http://en.wikipedia.org/wiki/Latin_square (latin squares)

Latin Squares:

http://en.wikipedia.org/wiki/Latin_square

http://mathworld.wolfram.com/LatinSquare.html

http://www.cut-the-knot.org/arithmetic/latin3.shtml

Sudoku:

http://en.wikipedia.org/wiki/Sudoku

http://en.wikipedia.org/wiki/Mathematics_of_Sudoku

http://www.websudoku.com

http://theory.tifr.res.in/ sgupta/sudoku

http://www.maa.org/editorial/mathgames/mathgames_09_05_05.html

http://www.geometer.org/mathcircles/sudoku.pdf

http://mathworld.wolfram.com/Sudoku.html