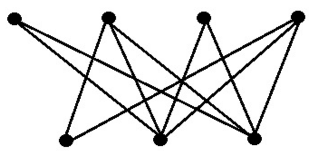
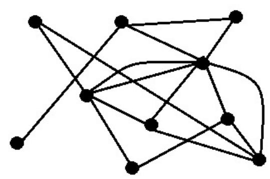
Math Teacher's Circle: Graph Theory

Basic graph theory

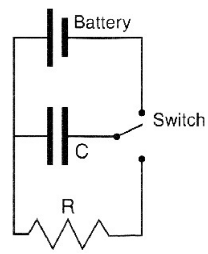
For each of the following graphs, determine:
A) The number of vertices,
B) The number of edges, and
C) The valency of each vertex.



1

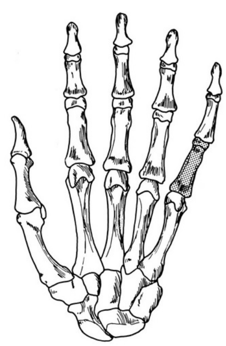


2

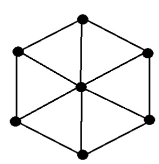


3

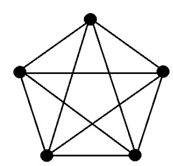
4: See below.



5: See below.



6: See below, this is an example of W_6



7: See below, this is an example of K_5

4: Take the graph whose vertices are the Northeastern states and whose edges correspond to states that share a border.
5: Take the graph whose vertices are the bones in a human hand and whose edges correspond to bones being connected by ligaments.
6: The wheel graph W_n which has n vertices in a regular n-gon, one more vertex v inside of it, and edges corresponding to the the polygon plus one edge connecting each vertex to v.
7: The complete graph K_n which has n vertices, and edges connecting every vertex to every vertex.

Brainstorm ways that graphs could encode useful information for 1) A biologist, 2) A chemist, C) A teacher, D) An administrator at a school, E) A historian, F) Other professions!

Map Coloring and Planarity

We can use graph theory to make map coloring more tractable. For each map,

A. Construct the corresponding graph as in example 4 above,

B. Determine the minimal number of colors needed (justify your answer!)

1

2

3

4) A map whose corresponding graph is K_n, for some n.
5) A map whose corresponding graph is W_n, for some n.

Notice that all these graphs are *planar*, that is, they can be drawn on the plane without having edges overlap at a non-vertex. Try to determine if the following graphs are or are not planar:

1) The graph given by all the latitude and longitude lines whose label is a multiple of $5°$. Here, the vertices are the points of intersection of all these lines.
2) Example 2 on the first page.
3) K_2, K_3, K_4, K_5, K_6.
4) The complete bipartite graph $K_{m,n}$ is a graph with $m + n$ vertices, broken into two sets M and N. Each vertex in M connects to every vertex in N, and thus each vertex in N connects to each vertex in M. No vertices in M connect to each other, similarly for N. Example 1 on the first page is $K_{3,4}$. Practice drawing some, and then try to determine if the following graphs are planar: $K_{2,2}$, $K_{2,3}$, $K_{3,1}$, $K_{3,3}$.

Euler's formula: If a planar graph has V vertices, E edges, and F regions (pieces that the graph cuts the plane into), then

$$V - E + F = 2$$

Can you use this to show that some of your "probably not planar" graphs are in fact *definitely* not planar?

Scheduling problems

Example: 6 people arrive late at a restaurant and discover that there is one serving left of each of seven dishes: pork chops, prime rib, vegetarian curry, trout,pasta with clam sauce, macaroni and cheese, and barbecued pork spare ribs. Alice cannot eat pork, Bob is vegetarian, Claire eats nothing that begins with the letter C, Dale is allergic to curry, clams and molasses, Ernie prefers pasta, but will settle for beef or fish, and Flo eats anything. Can everyone eat something?

Problems: First, try to answer the question. Then, see if graph theory can help with this problem.

One more problem of this sort:
An accounting firm with 8 CPA's available and 7 audits to complete needs to schedule their audits. The following chart shows whether the accountants are qualified to complete these audits (X = qualified). Is there a way to complete all these audits without pairing anyone to more than one audit?

Auditor:	1	2	3	4	5	6	7
a			X	X	X		
b	X			X		X	
c			X			X	X
d		X	X	X			
e	X	X	X				X
f		X		X		X	
g		X		X	X		
h				X	X	X	

Euler walks and circuits

A *walk* in a graph is a collection of vertices $v_1,\ v_2, \ldots,\ v_n$ so that there is an edge that connects v_i to v_{i+1} for each i in $1 \leq i \leq n$. A *circuit* in a graph is a walk that begins and ends at the same vertex, that is, $v_1 = v_n$. A circuit/walk is an *euler* circuit/walk if it uses every edge exactly once. Thus the warm-up problem asked you to find if the corresponding graph has an euler walk or an euler circuit.

A. Take a look at the corresponding graphs to the warm-up pictures (they're below). The graph which has an euler circuit has a very strong property. Can you determine what it is? The other graphs, which don't have an euler circuit, shouldn't have this property.

B. Can you prove that every graph that has an euler circuit has this property?

C. Try to formulate a similar property for graphs that contain an euler walk but **no** euler circuit.

D. *Challenge Problem:* Suppose it is known that graphs with the property in part B have euler circuits (that is, the *converse* of B is true). Can you use this to prove that graphs with the property in part C have an euler walk? (That is, if the converse of B is true, then the converse of C is true.)