Pick's Theorem

Based on material found on NCTM *Illuminations* webpages
Adapted by Aimee S. Johnson and Joshua M. Sabloff
version for leaders

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Content	Process
Representation of Functions	Using examples to find a pattern
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1   Goal

Imagine you have a board with a square pattern of pegs, each one unit away from the other. (That is, you have a partial representation of $\mathbf{Z^2}$). We'll use paper with dots on it as an easier-to-produce version. Imagine placing a rubberband around some of the pegs to create some type of figure (in our easier-to-represent version, you will connect the dots in such a way that you create a closed loop that doesn't cross itself). The rubberband will touch some number of the pegs (i.e. the drawn loop will go through some number of dots) : call these the **perimeter** pegs. Note that the figure will enclose some number of pegs/dots : call these the **interior** pegs.

The main goal: find a relationship between the area of the figure, the number of perimeter pegs, and the number of interior pegs.

2   Examples

2.1   First Example

Suppose you draw lines between the pegs at locations $(1, 0)$, $(0, 1)$, and $(2, 2)$. This gives a triangle, and we have: 3 perimeter pegs (i.e. $P = 3$), 1 interior peg (at $(1, 1)$, so $I = 1$), and an area of $A = 1.5$. How to find this area? If you think of the 2×2 square, which has total area 4, then can see that at the lower left we have left out area $\frac{1}{2}$, at the upper left we have left out area

$\frac{1}{2} \cdot 1 \cdot 2 = 1$, and at the lower right we have left out area 1, thus leaving the enclosed area to be $4 - 2.5 = 1.5$.

2.2 Another Example

If we consider the 3×3 box and draw lines between $(0, 1)$, $(0, 2)$, $(1, 3)$, $(2, 2)$, $(3, 3)$, $(3, 0)$, $2, 1)$, $(1, 0)$, and then back to $(0, 1)$, we get fish-like figure. This is a bit more interesting!

3 Answer

Pick's Theorem says

$$A = \frac{1}{2}P + I - 1.$$

4 Hints

First, groups should try a bunch of examples.
Guess and Check

If they guess the relationship is linear, then they could note that once they get 3 data points they can solve for the variables: thinking of this as $area = b \cdot (P) + c \cdot (I) + d$, one could even switch this into matrix form if wanted.

Rates of Change

But another possibility is to figure out the coefficients by seeing how changing the number of perimeter pegs changes the area. For instance, have the groups think about the following:

Construct a figure with $P = 4$ and $I = 0$. What is the area?
Construct a figure with $P = 6$ and $I = 0$. What is the area?
Construct a figure with $P = 8$ and $I = 0$. What is the area?

Now conjecture a rule based on these examples, and say what that would mean in the equation.

Similarly, create two figures with the same number of perimeter pegs but with different numbers of interior pegs. Find the area of each and make a conjecture for a rule about how increasing I increases A. What does this mean in the equation?

5 History

This theorem was presented by Austrian mathematician Georg Pick in 1899. However, it was only in 1969, that the theorem was brought to broad attention through the popular 'Mathematical Snapshots' by H. Steinhaus.

6 Extensions

6.1 What if there are holes?

Does the equation still hold if the shape has a hole in it? For instance, say one puts the rubberband around indices $(0,0)$, $(3,0)$, $(0,3)$, and $(3,3)$ (i.e. one makes a box of size 3×3), and then places another rubberband around indices $(1,1)$, $(2,1)$, $(1,2)$, and $(2,2)$ (i.e. around the inside box of size 1×1). Then what happens? Of course, one can compute the area of the whole thing and then subtract the area of the inside, but is there a more efficient way to do this? That is, is there an analog of Pick's Theorem here? Note that in this example, $P = 16$ and $I = 0$.

Answer: If there are h holes with $h+1$ boundaries (each a non-intersecting closed curve), then we get

$$A = \frac{1}{2}P + I + h - 1.$$

6.2 What about higher dimensions?

If we built 3-dimensional figures on $\mathbf{Z^3}$ and counted the number of pegs on the surface and the number of pegs in the inside, could we use that in some way to find the volume?

Hint: have them consider the pyramid figure going through $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(1,1,r)$ where r is an integer. (This is called a "Reeve tetrahedrom"). Note that it doesn't matter what r is: you'll always get that $P = 5$ and $I = 0$. But clearly you get different volumes. (Volume of a tetrahedron is $\frac{1}{3}$ (area of base) height). So evidently knowing P and I isn't enough.

7 Application

Pick's theorem can be used to answer the following:

Question: can you draw an equilateral triangle whose vertices lie on lattice points?

They should try, remembering that the base doesn't need to be horizontal.

But the point is that Pick's Theorem tells us the answer is no, because if you could then we could find the area of this triangle by the formula and we would get an integer multiple of $\frac{1}{2}$. But the area of an equilateral triangle with base s is $\frac{s^2\sqrt{3}}{4}$. Since s would be an integer in this case, the area must be an irrational.

8   References

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