I Walk the Line(s)

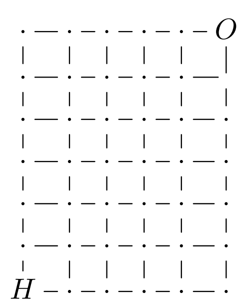
Aimee S. Johnson and Joshua M. Sabloff
version for leaders

August 13, 2013

Supplies: Large-scale graph paper (print this out)

1 The Problem: First Version

Imagine that you live in the city of Cartesia whose streets lie on the lines of a square grid. The founders of the city did not believe in creative names for the streets; instead, all north-south roads are numbered "avenues" (from south to north) and all east-west roads are numbered "streets" (from west to east). Your regular commute begins at your house at the (convenient) corner of 0^{th} street and 0^{th} avenue and ends at your office at the corner of 5^{th} street and 6^{th} avenue. You have been making this trip for years, but you are the restless (or adventurous) type, and you try to take a different route each day. On the other hand, you are not inefficient — you only move north and east. At some point, you start to wonder how long it will take you to try *all* of the routes.



Question 1. How many different ways are there to get from your house to your office?

Question 2. If your office moves to the corner of 7^{th} and 11^{th}, how many different ways are there to make your trip? (Can you ask and answer a more general question?)

2 Hints

The first thing to do is to try a smaller problem! Draw pictures! Organize data!

Some good guiding questions: have you seen this situation before (when doing a slightly larger problem)? How do you know you have *all* of the paths? Do you see any symmetry?

Later on, you might suggest working backwards by labeling the vertices of a grid with the number of paths there are from H to the grid point. Participants might recognize Pascal's triangle (or a piece of it)!

You could also suggest thinking about the total number of hops each path must take (11), and then thinking of the paths as sequences of H (for horizontal) and V (for vertical) segments, with 5 H's and 6 V's. (So you're really picking out the positions from 1 to 11 of, say, 5 places to go horizontal ... which is $C(11,5)$. Participants will probably not know this notation, however, and one could lead them through it.) I wouldn't expect many groups to solve the problem this way.

The answer to the first question is $C(11,5) = 462$, while the answer to the second question is $C(18,7) = 31,824$, while the answer to the general problem with the office at (x,y) is $C(x+y,x) = C(x+y,y) = \frac{(x+y)!}{x!y!}$.

3 The Problem: Second Version

Your office moves again, this time at 6^{th} and 6^{th}, back near its original location. Unfortunately, a horde of zombies invades Cartesia and occupies all of the blocks pictured below:

```
Z - Z - · - · - · - O
|   |   |   |   |   |
Z - Z - · - · - · - ·
|   |   |   |   |   |
· - · - · - · - · - ·
|   |   |   |   |   |
· - · - · - · - · - ·
|   |   |   |   |   |
· - · - · - · - Z - Z
|   |   |   |   |   |
H - · - · - · - Z - Z
```

Question 3. How many different ways are there to get from your house to your office while avoiding the zombies?

Question 4. If the horde of zombies shifts slightly and occupies the blocks below, how does your answer change?

```
Z – Z – Z — · — · — O
|   |   |   |   |   |
Z – Z – Z — · — · — ·
|   |   |   |   |   |
· — · — · — · — · — ·
|   |   |   |   |   |
· — · — · — · — · — ·
|   |   |   |   |   |
· — · — · — Z – Z – Z
|   |   |   |   |   |
H — · — · — Z – Z – Z
```

Either make up and solve your own question involving zombies or consider the following:

Question 5. The zombies decide to make your life extremely challenging — they now occupy the blocks below. How does your answer change? What if your office moves to 7^{th} and 7^{th}? More generally?

```
Z – Z – Z – Z – Z – O
|   |   |   |   |   |
Z – Z – Z – Z — · — ·
|   |   |   |   |   |
Z – Z – Z — · — · — ·
|   |   |   |   |   |
Z – Z — · — · — · — ·
|   |   |   |   |   |
Z — · — · — · — · — ·
|   |   |   |   |   |
H — · — · — · — · — ·
```

4 Hints

For the first problem, note that all paths must go through either the dot at $A = (4, 3)$ or $B = (3, 4)$ (and no path goes through both). Thus, one can split up the problem into counting paths from $H \to A \to O$ (which is $C(5, 2) \cdot C(5, 2) = 100$) and paths from $H \to B \to O$ (same number), with the total number of paths being the sum (200). "Work backwards" could help out here. The "labeling" strategy would be nice here and in the next one.

It's OK if nobody gets to the last question, but if they do ... the last question is asking participants to derive the Catalan numbers! The answer is:

$$C_n = \frac{1}{n+1} C(2n, n)$$

(In particular, $C_6 = 42$ and $C_7 = 132$). These show up in so many ways, it's ridiculous. For example: the number of ways to split a $n+2$-gon into triangles by connecting vertices with straight lines, or the number of ways to add parentheses to a string of $n+1$ letters like $(ab)(cd), a((bc)d), a(b(cd))$, etc., or ....

5 Bigger Picture

These problems come from the field of *combinatorics*, in which mathematicians study methods for counting (in interesting and perhaps unexpected ways) the number of elements in a (simply defined) finite set.

6 Common Core Standards

Process Standards:

MP1: Make sense of problems and persevere in solving them. (Of course, all of our sessions reference this standard!)

MP2: Reason abstractly and quantitatively.

MP3: Construct viable arguments.

MP5: Use appropriate tools strategically. (Graphical enumeration and models at the beginning, at least; tables of data; eventually, some algebra?)

MP8: Look for and express regularity in repeated reasoning. (This is what the "do a smaller problem" suggestion is really saying)

The content standards to which this activity gets close are:

5.G.A Graph points on the coordinate plane to solve real-world and mathematical problems.

7.SP.C.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. *Question 4 is a compound event question, in disguise.*

7 Sources

- The Math Forum, "Counting Possible Paths."
  `http://mathforum.org/library/drmath/view/60784.html`
- Wikipedia on Catalan numbers.