Teachers' Circle Workshop
Place Value Problems

In this session, we'll learn how to solve problems related to **place value**. This is one of the fundamental concepts in arithmetic, something every elementary and middle school mathematics teacher should understand profoundly. You already know that the digits used to represent a positive integer have different meanings depending on their position. This is commonly called *place value*.

Example 1. Pick a three digit number. Multiply it by 7. Then multiply your answer by 11, and finally multiply by 13. Explain why you got that answer.

Example 2. Next, consider the following problem. Find a four-digit number $\underline{abcd}$ which is reversed when multiplied by 9. In other words, find digits a, b, c, and d such that

$$9 \cdot \underline{abcd} = \underline{dcba}.$$

Solution. Our solution method is to reason digit by digit. First note that $a = 1$ since otherwise $9 \cdot \underline{abcd} \geq 9 \cdot 2000 = 18000$, which is a five-digit number. Since $a = 1$, it follows that $d = 9$. Now the equation take the following form:

$$9 \cdot \underline{1bc9} = \underline{9cb1}.$$

We can express this in decimal notation (in contrast to the underline notation we have been using) as follows:

$$9 \cdot (1009 + 100b + 10c) = 9001 + 100c + 10b.$$

Distributing the 9 across the $\cdot$ yields

$$9081 + 900b + 90c = 9001 + 100c + 10b$$

from which it follows that

$$80 + 890b = 10c.$$

Since the right side $10c$ is at most 90 (c is a digit), we can conclude that $b = 0$, and hence $c = 8$. Therefore $\underline{abcd} = 1089$ is the only such number. In the exercises, you will be asked to investigate larger numbers that reverse when multiplied by 9. In the next example we deal with several digits at once.

Example 3. Find a 6-digit number $\underline{abcdef}$ that becomes 5 times as large when the units digit f is moved to the left end of the number. In other words, solve $5 \cdot \underline{abcdef} = \underline{fabcde}$.

Solution. Before we solve this, let's consider how a six-digit number changes when the rightmost digit is moved to the left end. Take 123456 as an example. Note that $612345 = 600000 + 12345$, whereas $123456 = 123450 + 6 = 12345 \cdot 10 + 6$. If we give the name x to 12345, a common technique in algebra, we can write

$123456 = 10x + 6$ and $612345 = 6 \cdot 10^5 + x$. What the hypothesis tells us is that $5(10x+6) = 6 \cdot 10^5 + x$. Of course 123456 does not satisfy the equation, but replacing *abcde* with x reduces the six variables to just two. Of course we don't know that $f = 6$ works so we need to solve

$$5 \cdot (10x + f) = f \cdot 10^5 + x.$$

Distributing the 5 and migrating we get $50x + 5f = 10^5 f + x$ which is equivalent to $49x = (10^5 - 5)f = 99995f$. Both sides are multiples of 7 so we can write $7x = 14285 \cdot f$. Now the left side is a multiple of 7, so the right side must also be a multiple of 7. Since 14285 is not a multiple of 7, it follows (from the Fundamental Theorem of Arithmetic, which we have yet to prove) that f must be a multiple of 7. Since f is a digit, it must **be** 7. And x must be 14285. This technique is one that you will use repeatedly in the next few weeks.

Example 4. The amazing number 1089. Pick a three-digit number like 742. Reverse it to get 247 and subtract the smaller of the two from the larger. Here we get $742 - 247 = 495$. Now take the answer and reverse it to get 594, then add these two to get $495 + 594 = 1089$. What did you get? Now compute the product of 1089 and 9. You get $1089 \cdot 9 = 9801$. Isn't that odd, multiplying by 9 had the effect of reversing the number. Is there a connection between these two properties?

Example 5. Suppose a, b, c and d are digits (ie, in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$), and the sum of the two four-digit numbers $\underline{abcd}$ and $\underline{dabc}$ is 6017. Find all four-digit numbers $\underline{abcd}$ with this property. Note that $\underline{abcd}$ is a four-digit number only if $a \neq 0$.

Recall the two functions *floor* ($\lfloor\rfloor$) and *fractional part* ($\langle\rangle$), defined by $\lfloor x \rfloor$ is the largest integer that is less than or equal to x, and $\langle x \rangle = x - \lfloor x \rfloor$. For example, $\lfloor \pi \rfloor = 3$ and $\langle \pi \rangle = \pi - 3$, $\lfloor -\pi \rfloor = -4$ and $\langle -\pi \rangle = -\pi - (-4) = 4 - \pi$. Also, notice that for any real number x, $x = \lfloor x \rfloor + \langle x \rangle$. For example, $\lfloor 3.15 \rfloor + \langle 3.15 \rangle = 3 + 0.15 = 3.15$.

Example 6. Let N be a four-digit number and let $N' = \langle N/10 \rangle \cdot 10^4 + \lfloor N/10 \rfloor$. Suppose $N - N' = 3105$. Find all possible values of N. Recall the two functions *floor* ($\lfloor\rfloor$) and *fractional part* ($\langle\rangle$), are defined by $\lfloor x \rfloor$ is the largest integer that is less than or equal to x, and $\langle x \rangle = x - \lfloor x \rfloor$. **Solution.** Let $N = \underline{abcd}$. Then $\langle N/10 \rangle = \langle abc.d \rangle = 0.d$, so $10000\langle N/10 \rangle$ could be written as $1000d$. On the other hand $\lfloor N/10 \rfloor = \lfloor \underline{abc.d} \rfloor = \underline{abc}$. In other words, $N' = \underline{dabc}$, just as in the problem above. The equation $10x + d - (1000d + x) = 3105$ is equivalent to $9x - 999d = 3105$, and dividing both sides by 9 yields $x - 111d = 345$. There are six solutions for $x \geq 100$:

$$
\begin{aligned}
d = 0 \quad &\Rightarrow \quad x = 345, N = 3450 \\
d = 1 \quad &\Rightarrow \quad x = 456, N = 4561 \\
d = 2 \quad &\Rightarrow \quad x = 567, N = 5672 \\
d = 3 \quad &\Rightarrow \quad x = 678, N = 6783 \\
d = 4 \quad &\Rightarrow \quad x = 789, N = 7894 \\
d = 5 \quad &\Rightarrow \quad x = 900, N = 9005
\end{aligned}
$$

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During the rest of the session, please work on the following problems. You cannot finish them, but perhaps you will have time to work on them later.

1. **X'ing digits.** Consider the number

$$N = 123456789101112\ldots 5960,$$

obtained by writing the numbers from 1 to 60 in order next to one another. What is the largest number that can be produced by crossing out 100 digits of N? What is the smallest number that can be produced by crossing out 100 digits of N?

2. Use the five digits $1, 3, 5, 7$ and 9 exactly once to build two numbers A and B such that $A \cdot B$ is as large as possible. Then build two numbers C and D such that $C \cdot D$ is as small as possible.

3. For each of the following problems, let $S(n) = n$ in case n is a single digit integer. If $n \geq 10$ is an integer, $S(n)$ is the sum of the digits of n. Similarly $P(n)$ is n if n is a positive single digit integer and the product of the digits of n otherwise. If there is no solution, prove it.

   (a) What is the smallest solution to $S(n) = 2005$. Express your answer in exponential notation.

   (b) How many five-digit numbers n satisfy $S(S(n)) + S(n) = 50$.

   (c) Find all solutions to $S(S(S(n))) + S(S(n)) + S(n) = 100$.

   (d) Find all solutions to $S(S(n)) + S(n) + n = 2007$.

   (e) (2007 North Carolina High School Math Contest) Find the sum $S(1) + S(2) + \cdots + S(2007)$.

   (f) Can both $S(a)$ and $S(a+1)$ be multiples of 49?

   (g) Find a number n such that $S(n) = 2S(n+1)$. For what values of k does there exist n such that $S(n) = kS(n+1)$?

4. Recall the two functions *floor* ($\lfloor\rfloor$) and *fractional part* ($\langle\rangle$), defined by $\lfloor x \rfloor$ is the largest integer that is less than or equal to x, and $\langle x \rangle = x - \lfloor x \rfloor$.

   (a) For each member x of the set S, $S = \{\pi, 1.234, -1.234, \frac{7}{3}, -\frac{7}{3}\}$, evaluate $\langle x \rangle$ and $\lfloor x \rfloor$.

   (b) Define another function f by $f(x) = x - 10\lfloor \frac{x}{10} \rfloor$. Find $f(x)$ and $f(\lfloor x \rfloor)$ for each x in S.

   (c) Let $g(x) = \lfloor \frac{\lfloor x \rfloor}{10} \rfloor - 10\lfloor \frac{\lfloor x \rfloor}{100} \rfloor$. Evaluate g at each of the members of S.

(d) Prove that for any number $x = 100a + 10b + c + f$, where a is a positive integer, b is a digit, c is a digit, and $0 \leq f < 1$, $g(x) = b$. In other words, $g(x)$ is the tens digit of x.

5. (a) Grab a calculator. Key in the first three digits of your phone number (NOT the area code)
   (b) Multiply by 80.
   (c) Add 1.
   (d) Multiply by 250.
   (e) Add the last 4 digits of your phone number.
   (f) Add the last 4 digits of your phone number AGAIN.
   (g) Subtract 250.
   (h) Divide number by 2.

   Explain why you get such an interesting answer.

6. The numbers $1, 2, 3, 6, 7, 8$ are arranged in a multiplication table, with three along the top and the other three down the column. The multiplication table is completed and the sum of the nine entries is tabulated. What is the largest possible sum obtainable.

$\times$	a	b	c
d			
e			
f			

7. A *special-8* number is one whose decimal representation consists entirely of 0's and 8's. For example 0.8808 and $0.\overline{08}$ are special numbers. What is the fewest special numbers whose sum is 1.

8. A *special-3* number is one whose decimal representation consists entirely of 0's and 3's. For example 0.3033 and $0.\overline{03}$ are special numbers. What is the fewest special numbers whose sum is 1.

9. A *special-7* number is one whose decimal representation consists entirely of 0's and 7's. For example 0.7707 and $0.\overline{07}$ are special numbers. What is the fewest special numbers whose sum is 1.

10. What is the largest 5-digit multiple of 11 that has exactly 3 different digits?

11. A two-digit integer N that is not a multiple of 10 is k times the sum of its digits. The number formed by interchanging the digits is m times the sum of the digits. What is the relationship between m and k?

12. A check is written for x dollars and y cents, both x and y two-digit numbers. In error it is cashed for y dollars and x cents, the incorrect amount exceeding the correct amount by \$17.82. Find a possible value for x and y.

13. Solve the alpha-numeric problem $\underline{abcd} \times 4 = \underline{dcba}$, where a, b, c and d are decimal digits.

14. The rightmost digit of a six-digit number N is moved to the left end. The new number obtained is five times N. What is N?

15. Repeat the same problem with the 5 changed to a 4. That is $4(\underline{abcdef}) = \underline{fabcde}$

16. Let $N = 1234567891011\ldots19992000$ be the integer obtained by appending the decimal representations of the numbers from 1 to 2000 together in order. What is the remainder when N is divided by 9?

17. Find three three-digit numbers whose squares end with the digits 444.

18. A six-digit number $N = abcdef$ has the property that $7(\underline{abcdef}) = 6(\underline{defabc})$, where both digits a and d are non-zero. What is N?

19. Let a, b, c, d, and e be digits satisfying $4 \cdot \underline{abcde4} = \underline{4abcde}$. Find all five of the digits.

20. Let a, b, c, d, e be digits satisfying $\underline{abc} \cdot a = \underline{bda}$ and $\underline{bda} \cdot a = \underline{cdde}$. What is $\underline{cdde} \cdot a$? A reminder about notation: the string of digits abc can be interpreted two ways, first as $a \cdot b \cdot c$ and secondly as $100a + 10b + c$. To distinguish these two interpretations, we use the underline notation $\underline{abc}$ for the latter of these.

21. Let $N = \underline{abcdef}$ be a six-digit number such that $\underline{defabc}$ is six times the value of $\underline{abcdef}$. What is the sum of the digits of N?

22. **Maximizing Products**

    (a) Using all nonzero digits each once, build two numbers A and B so that $A \cdot B$ is as large as possible.

    (b) Using all nonzero digits each once, build three numbers A, B and C so that $A \cdot B \cdot C$ is as large as possible.

    (c) Using all nonzero digits each once, build four numbers A, B, C and D so that $A \cdot B \cdot C \cdot D$ is as large as possible.

    (d) If we build five two-digit numbers using each of the digits 0 through 9 exactly once, and the product of the five numbers is maximized, find the greatest number among them.

23. **Calling All Digits**

    (a) Using each nonzero digit exactly once, create three 3-digit numbers A, B, and C, such that $A + B = C$.

    (b) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $1 : 3 : 5$.

    (c) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $1 : 2 : 3$.

    (d) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $4 : 5 : 6$.

    (e) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $3 : 7 : 8$.

    (f) Are there any more single digit ratios $a : b : c$ for which the nine nonzero digits can be used to build three numbers A, B, and C in the ratio $a : b : c$.

    (g) Using the ten digits each exactly once, create three numbers A, B, and C, such that $A + B = C$.

24. The number $N = 123456789101112\ldots999$ is formed stringing together all the numbers from 1 to 999. What is the product of the 2007^{th} and 2008^{th} digits of N?

25. The number $N = 37! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 37$ is a 44-digit number. The first 33 digits are $K = 137637530912263450463159795815809$. In fact, $N = K \cdot 10^{11} + L \cdot 10^{8}$, where L is less than 1000. Find the number L.

26. Find the smallest integer multiple of 84 and whose decimal representation uses just the two digits 6 and 7.

27. (Mathcounts 2009) Find a six-digit number $\underline{abcdef}$ such that $4 \cdot \underline{abcdef} = 3 \cdot \underline{defabc}$.

28. Find the greatest 9-digit number whose digits' product is 9!.

29. Find values for each of the digits A, B and C.

$$\begin{array}{r} BA \\ AB \\ \underline{+AB} \\ CAA \end{array}$$

30. The digits of $S = 2^{2008}$ are written from left to right followed by the digits of $T = 5^{2008}$. How many digits are written altogether? We can build some notation to simplify the solution and to help us think about the problem. Let $x||y$ denote the concatenation of integers x and y. Thus $2^4||5^2 = 16||25 = 1625$.

31. The number 2^{29} contains nine digits, all of them distinct. Which one is missing?

32. **Great Numbers.** Call a nine-digit number *great* if it consists of nine different nonzero digits. It is possible that two great numbers can have a great sum. In fact, it is even possible that two great numbers can both have a great sum and also be identical in seven places. Find two such numbers.

33. **Finding the unknown digit.**
Let $N = \underline{abcde}$ denote the five digit number with digits a, b, c, d, e and $a \neq 0$. Let $N' = \underline{edcba}$ denote the reverse of N. Suppose that $N > N'$ and that $N - N' = 670x3$ where x is a digit. What is x?

34. Recall also that the remainder when a decimal integer is divided by 9 is the same as the remainder when the sum of its digits is divided by 9. For example $1234 = 9k + 1$ since $1 + 2 + 3 + 4 = 10 = 9 + 1$.

    Let $a_1 = 1$ and for each $n \geq 1$, define a_{n+1} as follows:

$$a_{n+1} = n + 1 + a_n \cdot 10^{1+\lfloor \log(n+1) \rfloor}.$$

    Let's work out the first few values of the sequence. For example,

$$\begin{aligned} a_2 &= 2 + a_1 \cdot 10^{1+\lfloor \log(n+1) \rfloor} \\ &= 2 + 1 \cdot 10^{1+\lfloor \log(2) \rfloor} \\ &= 2 + 1 \cdot 10^{1+0} \\ &= 2 + 1 \cdot 10 = 12 \end{aligned}$$

    and a_3 is

$$\begin{aligned} a_3 &= 3 + a_2 \cdot 10^{1+\lfloor \log(2+1) \rfloor} \\ &= 3 + 12 \cdot 10^{1+\lfloor \log(3) \rfloor} \\ &= 3 + 12 \cdot 10^{1+0} \\ &= 3 + 12 \cdot 10 = 3 + 120 = 123. \end{aligned}$$

    Using the definition, we can show that $a_4 = 1234$, etc. This definition is said to be *recursive*. Recursion is a popular topic in discrete mathematics. We'll see later that the Principle of Mathematical Induction is an important tool in reasoning about recursively defined sequences.

    (a) How many digits does a_{2008} have?

    (b) Find the smallest n such that a_n has at least 2008 digits.

    (c) How many 1's are in the representation of a_{2008}?

    (d) How many 0's are in the representation of a_{2008}?

    (e) Find the first 5 multiples of 9 among the a_n.

    (f) Find the remainder when a_{2008} is divided by 9.

    (g) What is the smallest n such that a_n is a multiple of 11?