Teachers' Circle Workshop
Problems with Cubes

Let n be a positive integer. Consider the $n \times n \times n$ cube. Paint the exterior of the cube. Then cut the cube into n^3 unit cubes. Let $N_0, N_1, N_2, \ldots, N_6$ denote the number of cubes with $0, 1, 2, \ldots, 6$ painted faces. Of course $N_6 = 0$ unless $n = 1$, and then $N_6 = 1$. It is not hard to see that, for each n, $\sum_{i=0}^{6} N_i = n^3$. For example, when $n = 3$, we have $N_0 = 1, N_1 = 6, , N_2 = 12, N_3 = 8$, and $N_4 = N_5 = N_6 = 0$.

1. Find the number of cubes with some painted faces as a function of n. Note that for $n = 3$, the number is $27 - 1 = 26$.

2. Suppose that every pair of interior faces are glued together so that each pair of faces requires one unit of glue. How many square units of glue is needed? Examine this for $n = 4, 5$, and 6.

3. What is the minimum amount of glue needed to hold together all n^3 cubes, for $n = 2, 3, 4, 5$. This brings up the issue of whether a cube is rigid if the surface is rigid. We can try this problem under both assumptions, one where we simply need make the surface rigid and the other where every cube must get some glue. Try these for $n = 3$ where only the surface of the cube need be rigid.

4. A 10×10 square is decomposed into exactly 75 squares of various (integer) sizes. How many 3×3 squares are in this decomposition?

5. What is the fewest cuts needed to separate a wooden $3 \times 3 \times 3$ cube into 27 unit cubes if you're allowed to move blocks of cubes about before cutting? What if the big cube is $4 \times 4 \times 4$?

6. Again, suppose all six faces of an $n \times n \times n$ are painted red. Then one of the n^3 unit cubes is randomly selected and tossed like a die. What is the probability that the face obtained is painted? Of course, your answer depends on n. Try this for $n = 1, 2$, and 3. Then make a conjecture and prove your conjecture.

7. Let $n = 3$. Randomly and simultaneously select two unit cubes and toss them. What is the probability that they both have painted faces showing?

8. Suppose two non-adjacent faces of the big cube are painted red and the other four faces painted black. Let R denote the number of unit cubes with some red faces and B the number of unit cubes with some black faces. Find an n for which $B - R = 390$.

9. (2008 Mathcounts) A $12 \times 12 \times 12$ cube is built using a $10 \times 10 \times 10$ cube and a bunch of $2 \times 2 \times 2$ cubes. How many $2 \times 2 \times 2$ cubes are needed? A

$(n+2) \times (n+2) \times (n+2)$ cube is built using a $n \times n \times n$ cube, a bunch of $2 \times 2 \times 2$ cubes and a few unit cubes. How many unit cubes and $2 \times 2 \times 2$ cubes are needed? Your answer may depend on the oddness or evenness of n.

10. A 10×10 square is decomposed into exactly n squares of various (integer) sizes for certain values of n. Let S denote the set of value of n for which such a decomposition exists. Then, for example $1, 4, 25$ and 97 all belong to S. What is the size of S?

11. You have an unlimited supply of red(R) and blue(B) faces out of which to build cubes. How many distinguishable cubes can you build? Next suppose you have three colors.

12. Bob and Ann play the following game with 8 white unit cubes. Ann wins if she can assemble a $2 \times 2 \times 2$ cube that has only white faces exposed. But Bob gets to paint four of the $8 \cdot 6 = 48$ white faces black. Who wins?

    (a) What is the fewest number of faces Bob can paint to deny Ann in the $3 \times 3 \times 3$ game?

    (b) What is the fewest number of faces Bob can paint to deny Ann in the $4 \times 4 \times 4$ game?

    (c) What is the fewest number of faces Bob can paint to deny Ann in the $2 \times 3 \times 4$ game?

13. You're given 8 unpainted cubes. Can you paint the faces with two colors, red and blue, so that when you're done, you can assemble both an all red cube and an all blue cube?

14. You're given 27 unpainted cubes. Can you paint the faces with three colors, red, white, and blue, so that when you're done, you can assemble an all red $3 \times 3 \times 3$ cube, an all white $3 \times 3 \times 3$ cube and an all blue $3 \times 3 \times 3$ cube?

15. Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. The number of unit cubes with some red paint is found to be exactly 200 larger than the number of cubes with some black paint. How many cubes have no paint at all?

16. Suppose some faces of an $n \times n \times n$ wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Let R denote the number of cubes with some red paint and B the number of cubes with some

black paint. What is the least value of n for which $B + R$ is a multiple of 100? Find the next five values of n for which $B + R$ is a multiple of 100. In each case decide how the faces of the big cube are painted.

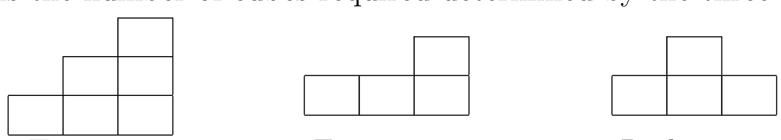
Using perspectives

17. This problem is about using cubes to build polyhedra that resemble buildings. Students can practice spacial visualization and use their imagination. Here's a sample problem. Find all possible cubical buildings that have a base, and front projection and a right side projection that is $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$

    Solution. There are seven solutions. We depict them using the base diagram where the number in each square represents the number of cubes on top of that square. For example $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 2 \\ \hline\end{array}$ is the solution that uses the maximum number of cubes, whereas $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline\end{array}$ uses the minimum number of cubes. How many solutions are there altogether? Answer: 7. Two solutions can be built with 6 cubes, four with 7 cubes and one with 8 cubes.

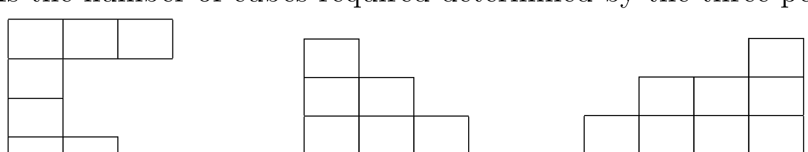
    For each problem below, build a model that is consistent with the given perspectives.

    (a) Is the number of cubes required determined by the three perspectives?

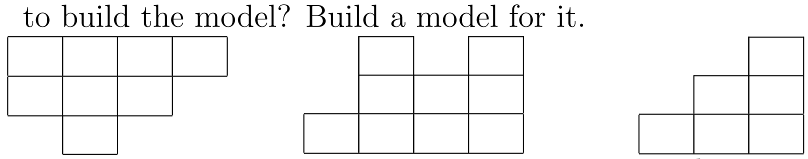
    Top view   Front view   Right view

    (b) Is the number of cubes required determined by the three perspectives?

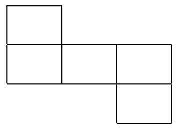
    Top view   Front view   Right view

    (c) Is the number of cubes required determined by the three perspectives? What are the maximum number and minimum number of cubes needed to build the model? Build a model for it.

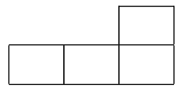
    

    Top view   Front view   Right view

    (d) The top and front projections are given. Build a possible right view. How many possible right views are there?

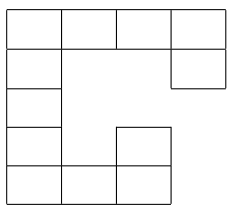


Top view

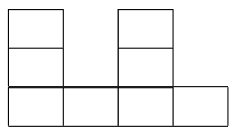


Front view

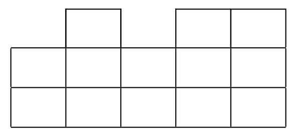
(e) Use exactly 20 cubes to make a model from the building plans below. Record the base plan for your building. What are the maximum and minimum numbers of cubes that could be used to build the structure.



Top view



Front view



Right view