San Joaquin Math Teachers' Circle (SJMTC)

Facilitator: Chris Goff

Title: The Circles of Apollonius (or: What's up with the logo?)

1. Math History I: The Problem of Apollonius (c.262 - c.190 BCE)

(See handout below.) Give the ancient problem that Apollonius solved: to find a circle that is tangent to three given circles. Can you find them all? How many such circles are there? How can you be sure?

2. A Word about Apollonius of Perga and Pappus of Alexandria (c.290-c.350 CE)

Brief talk about who Apollonius was, who Pappus was, and how we even know about this problem.

3. Look at the image from the "logo" for the SJMTC. What do you notice? What do you wonder about?

(See handout below.) [Teachers should notice there is one large circle, and several smaller circles contained within it. The smaller circles are tangent to one another. Each small circle has an integer in it: 2, 3, 6, 11, etc. The numbers get larger as the circles get smaller. They may also notice the symmetry. Internal tangency (one circle inside the other) versus external tangency (neither circle inside the other).]

[Possible questions: What do the numbers mean? How are these circles formed? Why start with two circles that have the number 2 on them? Are other circles possible? Does the symmetry have to be there?]

4. Math History II: A letter of Descartes (1643)

Talk about Descartes' response to Princess Elizabeth of Bohemia, in which he finds the radius of a circle that is tangent to three given mutually tangent circles. His formula can even be used to find the circles that are internally tangent to given circles, etc.

Triumph of (his) analytic geometry, which he knew, but really too long (and hard!) for us to go over its derivation. Descartes didn't even show all his work in his letter to the princess. But he did find a really nice way to write down the answer.

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

where k_j is the curvature of circle j, defined as $\pm\frac{1}{r_j}$, where r_j is the radius of circle j, and the sign is determined by whether the tangency is internal or external.

5. Apollonian Gaskets (cf Wikipedia)

Explain how such a gasket is drawn.

If you start with two circles of radius ½ inside a circle of radius 1, then what is the radius of the circle that is tangent to all three? (Actually, there are two such circles.) What is the radius of the next circle in the chain? Do a few more.

Can you create another gasket (on paper)? Hint: try starting with one circle of radius 1/3, and putting two circles of radius 1/8 inside it.

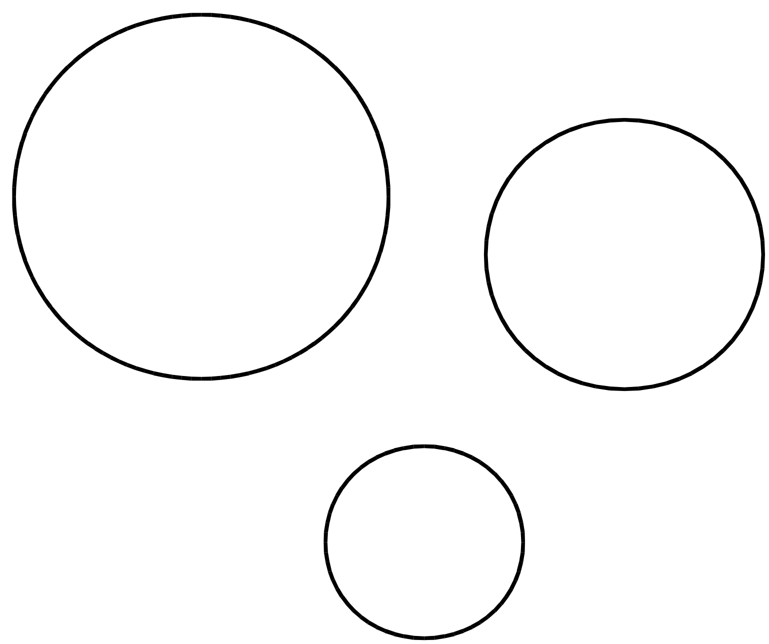
Try a radius of 1/6, and put circles of radius 1/10 and 1/15 inside it.

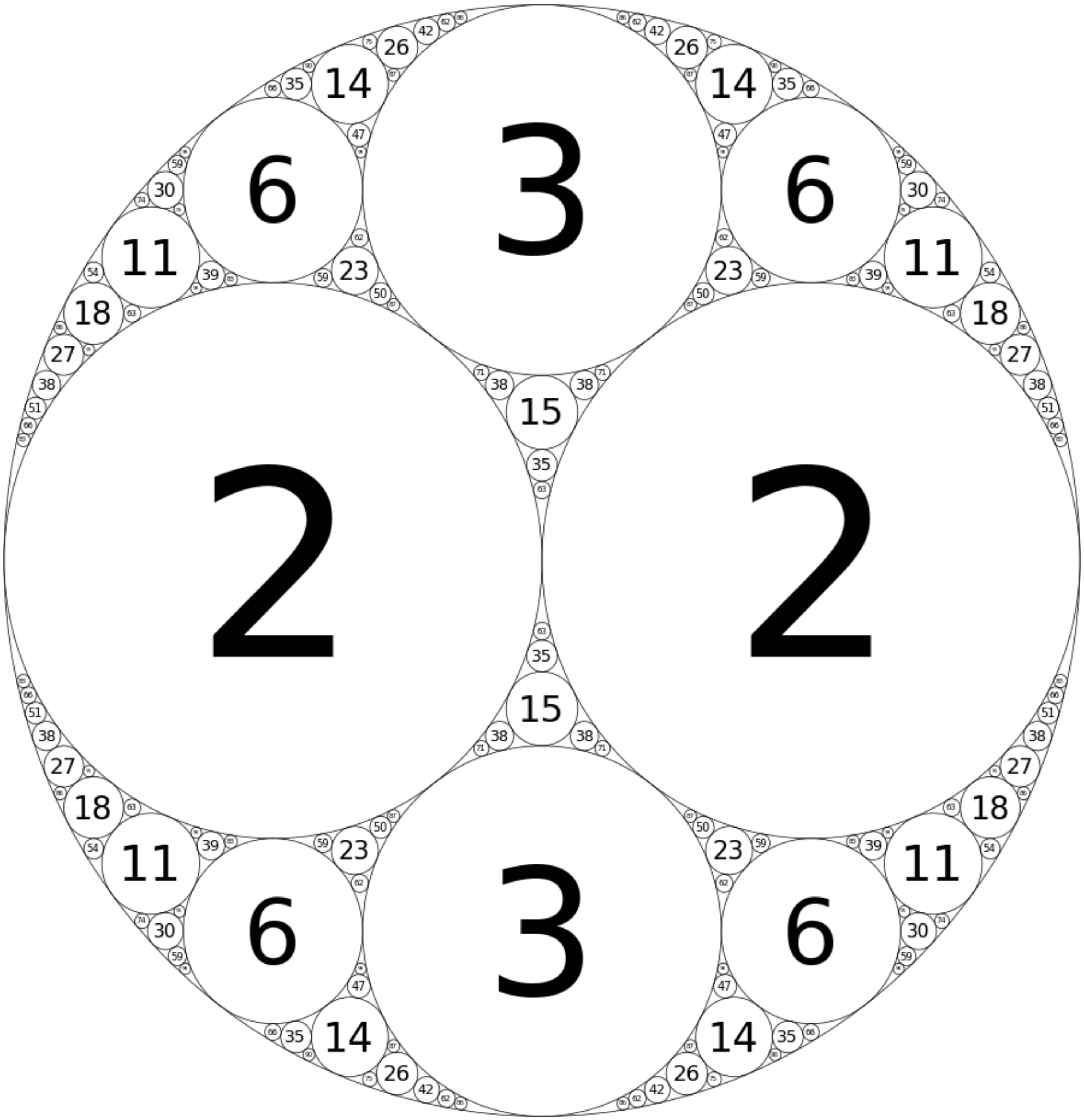
Do they all work out nicely? What does "work out nicely" mean here?

6. Possible Extensions
    a. What if one circle is a line? The formula actually gets simpler. How? (Hint: think of its curvature.)
    b. What if you have circles of radius 1 that are tangent to a number line at the integers? If you fill in the circle that is between two such circles (say the ones touching at 0 and at 1), and still tangent to the number line, then that circle touches the number line at what value? Where do the next circles touch? Continue to get "Ford" circles.

Find a circle that is tangent to all three given circles.

How many such circles can you find?





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