After the Jug Band

Warm up problem. After the Dynamic Jug Duo finish playing a song, Joe asked if the audience had any requests. "Yeah" came a shout from the audience. "Using just that 5 pint jug and that there 12 pint jug measure me 1 pint of water!" Is this possible with just the two jugs? How?

What volumes can be measured using only the 5 pint and 12 pint jugs? Obviously we cannot get more than 17 pints, and 17 is possible if we fill both jugs. But can we get 16 pints? 9? 2? What if we had jugs of different size? What if we had three jugs?

1. Using the 5 and 12 pint jugs can you get exactly 16 pints? 9? 2? Are there any quantities between 1 and 17 pints that you cannot get? How do you know?

2. Suppose you have jugs of 15 and 36 pints. What volumes between 1 and $15+36=51$ pints can you get? Which can't you get and why?

3. One way to get 2 pints with the 12 and 5 pint jugs is to fill the 12 pint jug, use it to fill the 5 pint container twice (emptying it each time) and then pouring the remainder in the 12 pint jug into the 5 pint jug. This gives you 2 pints in the 5 pint jug. Now, leaving this 2 pints in the smaller jug, repeat the process, filling the 12 pint jug, using it to fill (then empty) the 5 pint jug as many times as possible, then pouring the residual into the small jug. How much is in the small jug now? What happens as you repeat these steps?

4. After the observation in Problem 3, tell how you can get any volume from 1 to 17 pints.

5. Suppose we have jugs of 17 and 7 pints. Come up with a repetitive process that leads to any volume from 1 to $7-1=6$ pints. How can you then get any volume from 1 to $17+7=24$ pints?

6. By carefully tracking the process in Problem 5, show how you can find integers m and n so that $17m-7n=1$. This shows that the greatest common divisor of 17 and 7 (e.g.1) can be written as a linear combination of 17 and 7. (But finding the integers m and n to achieve this can be tedious!)

7. Review the amounts that emerged, in order, when performing the repetitive processes in Problems 3 and 5. What pattern do you notice in each of these sequences of numbers? How can these sequences be produced more quickly?

8. Now suppose we have two jugs, one of p pints and another of q pints with $0 < p < q$, and suppose p and q are relatively prime (i.e., have only 1 as a common divisor.) Suppose when we divide q by p we get a remainder of r. How can you get exactly r pints of water? If you now repeat the process that lead to r pints, what other amounts can you get? Formulate a good description of the repetitive process you used.

9. Referring to Problem 8, will the repetitive process lead to a solution for exactly one pint? How does this lead to a way to write 1 as a linear combination of p and q?

10. For each of the following pairs of integers, first find the greatest common divisor of the pair. Then imagine you have a jug of each size. In each case, tell how to get exactly the GCD amount of water using the two jugs, and write the GCD as a linear combination of the two numbers.

    a. 28, 49

    b. 13, 21

After the Jug Band—Notes

1. For the 5 and 12 pint jugs the key is to find ways to get each of 1, 2, 3, 4 pints. Once these can be obtained, all other volumes up to 17 pints can be obtained. For example to get 11 pints, first get 1 pint in the 5 pin jug and pour it into the 12 pint jug. Then pour two more full 5 pint jugs into the 12 pint jug. To obtain the smaller amounts, we repeat the following process:

   > Fill the 12 pint jug and use these 12 pints to repeatedly fill or top off the 5 pint jug... each time the 5 pint jug is filled to the top, empty it. Keep this up until the last pour from the 12 to the 5 does not fill the 5.

   The first time we do this process we end up with 2 pints in the 5 pint jug. (Note 2 is the remainder when 12 is divided by 5.) The next time we do the process there are 2 pints in the 5 pint jug when we start. We top off the 5 pint jug (this takes 3 pints from the 12 pint jug for the first pour) and eventually we end up with 4 pints in the 5 pint jug. (Note 4 is the remainder then $2 \cdot 12 = 24$ is divided by 5.) Run the process again to get 1 pint (the remainder when $3 \cdot 12 = 36$ is divided by 5) and a fourth time to get 3 pints (and 3 is the remainder the $4 \cdot 12 = 48$ is divided by 5.)

   To summarize, we see that by repeating the process 4 times, we can get, as amounts, each of the remainders when $1 \cdot 12$, $2 \cdot 12$, $3 \cdot 12$, $4 \cdot 12$ is divided by 5. Note these remainders are all different and give us amounts of 2, 4, 1, 3 pints. Note too that the process of getting 1 pint involved filling the 12 pint jug 3 times, and emptying the 5 pint jug 7 times. This can be summarized as

   $$1 = 12 \cdot 3 - 5 \cdot 7.$$

   Among other things, this demonstrates how the GCD of 5 and 12 (e.g. 1) can be written as a *linear combination* of 5 and 12. This illustrates the following result

   > Let m and n be positive integers with GCD of d. Then there are integers x and y such that
   > $$d = m \cdot x + n \cdot y.$$

   To produce this linear combination we just go through the jug procedure with jugs of m and n pints, keeping track of how many times the large jug is filled and the smaller one is emptied in arriving at d pints.

2. Notice that this is really Problem 1 again. First observe that since both 15 and 18 are multiples of 3, we can only obtain volumes that are multiples of 3. However, we can get every such volume from 3 to 33. One quick way to see this is to define a new unit of measure the thpint, (1 thpint is equal to 3 pints.) Our problem now involves jugs of 5 and 12 thpints. This is Problem 1.

8. This problem pulls together the ideas suggested in problem 1-7. By repeating the process described above (topping off the smaller jug with the larger jug) we obtain in the smaller jug, the following sequence of volumes:

$$\begin{aligned}
r_1 =& \text{remainder when } 1 \cdot q && \text{is divided by } p \\
r_2 =& \text{remainder when } 2 \cdot q && \text{is divided by } p \\
r_3 =& \text{remainder when } 3 \cdot q && \text{is divided by } p \\
&\vdots \\
r_{p-1} =& \text{remainder when } (p-1) \cdot q && \text{is divided by } p
\end{aligned}$$

Amazingly, these remainders are all different! To see this, assume two from this list, say $j \cdot q$ and $k \cdot q$ have the same remainder on division by p, and assume $1 \leq k < j \leq p - 1$. This means that

$$jq - kq = (j - k)q$$

is also a multiple of p. Since p and q are relatively prime, they can have no factors in common. This means p must be a factor of $j - k$. But this is impossible because because $0 < j - k < p$. Because all of the remainders are different and none of the numbers q, $1 \cdot q$, $2 \cdot q, \ldots, (p-1) \cdot q$ is a multiple of p, these remainders must be the numbers $1, 2, \ldots, p - 1$ is some order.