Shuffling, Cycles, and Matrices

Warm up problem. Eight people stand in a line. From left to right their positions are numbered 1, 2, 3, . . . 8. The eight people then change places according to THE RULE which directs a person in one position to another. For example, the person in position 1 moves to position 5, and we denote this by $1 \rightarrow 5$. Here is THE RULE in its entirety

$$1 \rightarrow 5, \quad 2 \rightarrow 2, \quad 3 \rightarrow 1, \quad 4 \rightarrow 7, \quad 5 \rightarrow 6, \quad 6 \rightarrow 3, \quad 7 \rightarrow 8, \quad 8 \rightarrow 4.$$

Suppose THE RULE is applied repeatedly. Will there ever be a time when every person is back in their original position? What is the minimum number of times before this happens? Can you think of other ways to represent THE RULE?

Many magicians are wizards with cards. The best of them can take a deck of 52 cards and repeatedly execute a perfect shuffle. In this way if they start with a new deck and do a couple of shuffles, they will know the position of every card in the deck. In fact, by executing enough perfect shuffles the deck can be returned to its original ordering. How many shuffles does this take?

Before answering this question, we need to decide what we mean by a "perfect shuffle." Not surprisingly there are many different ways to shuffle cards. Today we will work with the *overhand shuffle*. Of course there are many other ways to define a shuffle.

We assume that our deck has an even number of cards, say $2n$. Divide the deck into two equal stacks, and place the upper stack (cards 1 through n) to the right of the lower stack (cards $n + 1$ through $2n$.) This is step b. in the figure below. Beginning with the right stack, alternately take the top card from each stack and place them in a new pile. Continue until no cards are left. See c. in the figure. Note that this can be done in with the riffle shuffle action we are all used to when shuffling cards, but the two step procedure described here is needed so we fully understand what kind of shuffle is being considered in this case.
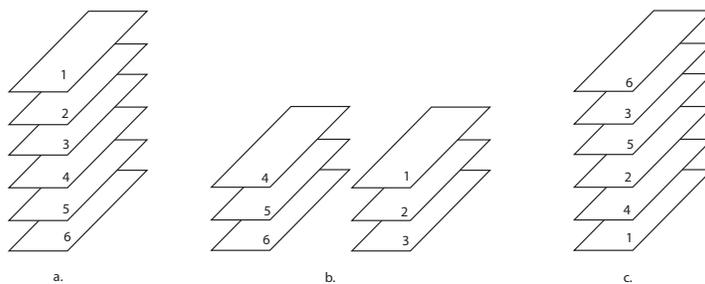
1. Find the number of overhand shuffles needed to return a deck to its original ordering. Do this for decks of 2, 4, 6, 8, 10, 12, 14, 16 cards. (The minimum number of shuffles needed is called the *period (or order) of the shuffle*).

A shuffle is a *permutation* (e.g., rearrangement) of the cards. Permutations can be described in many ways. One way is the arrow notation seen in the warm-up problem. This permutation can also be described by the structure

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 1 & 7 & 6 & 3 & 8 & 4 \end{pmatrix}$$

A permutation can also be represented by describing its cycles. What do you suppose is meant by this?

2. Write out the cycle structure for each of the shuffles in Problem 1. How does this help you find the period of the shuffle? What is the general result here linking periods and cycles?

3. Write out the card stack diagram (as in c. of the figure) and then the cycle diagram for the overhand shuffle of a 52 card deck. How many overhand shuffles does it take to return a 52 card deck to its original ordering?

4. Suppose you have a deck of $2n$ cards and do an overhand shuffle. To what position does card k go if $1 \le k \le n$? Where does it go if $n + 1 \le k \le 2n$?

5. Suppose you have a deck of 64 cards and you do an overhand shuffle. Where does card 7 (e.g., 7 from the top) end up? Where will this card be after a second overhand shuffle? After the third? Fourth? Keep going until you get back to the original position. How many steps did it take? What does this tell you about the period for a 64 card deck?

6. In the 64 card deck there are two cards that always stay in the same position. What are they?

7. Invent an overhand shuffle for a deck with and odd number (say $2n + 1$) of cards. Calculate the period for some small decks.

A permutation can also be represented by its *incidence matrix*. The matrix for the warm-up permutation is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where, for example, the 1 in the fifth row, sixth column, indicates that the person in position 5 goes to position 6.

8. Write out the incidence matrix for an overhand shuffle of 6 cards. Now do two consecutive overhand shuffles on the six card deck. What is the incidence matrix for the two shuffle permutation? What is the relationship between these two matrices? What would be the incidence matrix for four overhand shuffles of a six card deck? How does this matrix relate to the other two?

Incidence matrices are also an important part of *graph theory*. A graph is a collection of points (or vertices) and edges that join pairs of these points. As an example, here is a picture of a graph with 5 vertices and 8 edges.
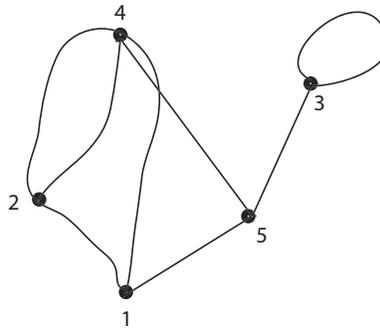
If we number the vertices, as in the figure, then we can write an incidence matrix for this graph:

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

where the 2 in row 4 and column 2 (and row 2 and column 4) indicates that there are two edges connecting vertices 2 and 4.

9. For each pair (a, b) of vertices count the number of ways to get from vertex a to vertex b on a path that is exactly two edges long. Write the incidence matrix for this set of data. How is this matrix related to M? Explain your reasoning. How could you get the number of paths at 3 edges long between each vertex?

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Shuffling, Cycles, and Matrices—Notes

There are two important ideas/skills underlying these activities. The first is the idea of a cycle. The second is matrix multiplication.

The cycles for a permutation can be found by picking out one element and tracing its "path" until it gets back to itself. For example, when the rearrangement for the warm-up problem is displayed as cycles we have:
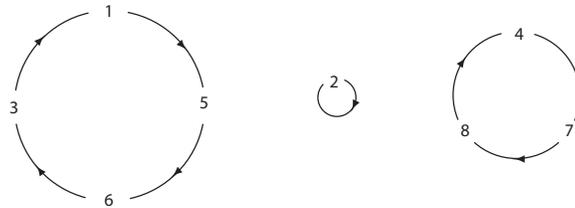
The first part of the cycle shows that the people is positions 1, 5, 6, 3 are back in their original positions after four (or eight, or twelve...) applications of THE RULE. This is a cycle of length 4. The number 2 is the only number in its cycle since the person in this position never moves. Finally, the numbers 4, 7, 8 are in a cycle of length 3, so the people in these positions return to their initial position after three (or six or nine...) applications of THE RULE. How many applications does it take for all people to be back in their original positions? The number must be a multiple of 4 and of 3, and we are lead to the LCM of 4 and 3 which is 12.

4. Gather some data: $1 \to 2n$, $2 \to 2n - 2$, $3 \to 2n - 4$,...,$n \to 2 = 2n - 2(n - 1)$. Thus

$$\text{if } 1 \leq k \leq n \text{ then } k \to 2n - 2(k - 1).$$

Similarly,
$$\text{if } n + 1 \leq k \leq 2n \text{ then } k \to 2(2n - k) + 1.$$

5. Using the formulae from the previous problem (but be careful!) we see 7 is in a cycle of length 6. Thus the period for the 64 card shuffle is a multiple of 6.

6. Again use the formulas in 4. If $1 \leq k \leq 32 = n$ then card k stays in the same position if
$$k = 64 - 2(k - 1),$$
from which $k = 22$. Similarly, using the other part of 4 we see card 43 stays in the same position with each shuffle.

8. The incidence matrix for two consecutive shuffles can be found by multiplying the one shuffle matrix by itself:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

As illustrated, the element in the 2nd row and 5th column is found by taking the *dot product* of the 2nd row of the first matrix with the 5th column of the second:

$$(0,\ 0,\ 0,\ 1,\ 0,\ 0) \cdot (0,\ 0,\ 0,\ 1,\ 0,\ 0) = 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 1.$$

The 1 in this position of the product indicates that after two shuffles, card 2 is in position 5.