Where Can I Park?

There are n people who live on Park Lane, a dead end street (no, it is *not* one way!) On the street, in a line, are n parking spaces on one side of the street, with no parking on the other side. The parking spaces are numbered $1, 2, \ldots, n$, with 1 closest to the street entrance, 2 the next closest, etc. Initially all of the spaces are empty. Each resident has a preferred parking place, perhaps close to his home or apartment, and it is possible that two different people have the same favorite space. Now suppose the people on the street arrive home in some order, $1, 2, 3, \ldots, n$, and that person i has space s_i as his favorite space. When person i arrives at the street, he drives down until he gets to his favorite space. If that is taken, he drives past it and takes the first unoccupied space after his favorite. Let $S = \{s_1, s_2, s_3, \ldots, s_n\}$ be the list of favorite spaces of the drivers in the order they arrive to park. We call S a (successful) parking sequence if everybody can find a parking place when they arrive in this order.

For example, if there are three people, then the sequence $S = \{2, 1, 2\}$ is a parking sequence, but the sequence $S = \{2, 2, 3\}$ is not. In the $\{2, 1, 2\}$ case, the first arrival parks in space 2 and the second in space 1. However, the third arrival find his favorite spot occupied so moves on to the next available spot, which is 3. Thus $S = \{2, 1, 2\}$ is a parking sequence because everyone gets a parking place, though some do not get their favorite place. In the $\{2, 2, 3\}$ case the first arrival takes space 2, then second takes 3, and the third, who drives to space 3 first but finds it occupied cannot park. Thus $S = \{2, 2, 3\}$ is not a parking sequence.

Parking places on Park Lane
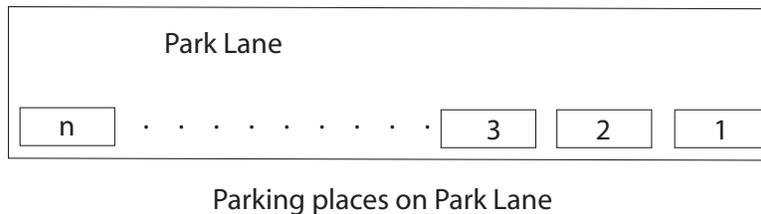
Warm-up Problem. Find all parking sequences for each of the cases $n = 1$, $n = 2$. $n = 3$. For the case $n = 4$ write down four sequences that are parking sequences and 4 that are not. What conjectures arise from these examples?

1. Lets look at a larger case, say $n = 12$ parking places and people. If we have a successful parking sequence, what can be said about the number of people who prefer space 12? Suppose we pull aside all people who prefer one of spaces 9, 10, 11, 12. What is the largest number of people that can be in this group? Can you convert this into a general rule about any parking sequence?

2. Again suppose we have a successful parking sequence. Must someone have place 1 as their favorite spot? Could there be more than one person who prefers space 1? Suppose that only one person prefers space 1. What more can you say about the parking sequence?

3. Let $\{s_1, s_2, \ldots, s_n\}$ be a parking sequence. Take the numbers in the sequence and rearrange them into nondecreasing order: $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ (so the a_j's are just the s_k's in a different order.) What do you notice about the number a_k in position k, for any k? How does this relate the the ideas discussed in Problem 1?

4. Suppose you have a (successful) parking sequence and rearrange it to get a nondecreasing sequence as in Problem 3. Is the rearranged sequence also a parking sequence? What if you have a sequence $a_1 \leq a_2 \leq \cdots \leq a_n$ with $a_i \leq i$ for each i. If we rearrange *this* sequence must the rearrangement also be a (successful) parking sequence?

5. Find a convincing argument that *any* permutation (e.g. rearrangement) of a parking sequence is also a parking sequence.

If we have n parking spaces and n drivers, how many successful parking sequences are there? To answer this question we alter the parking problem a bit. Suppose now that there are $n + 1$ spaces on a one-way round-a-bout. Again, n cars want to park, and again each driver has a preferred parking space, but now they have the luxery of not running into a brick wall at the end of the street—they can continue on the roundabout until they find an empty space, even if this this means starting on a second trip. Note that when all drivers have parked there will be one empty space.

We now look at the connection between the one-way street parking problem and the round-about parking problem. We can slip from one problem to the other in this way. Given a one-way street parking situation for n cars, we can add a space number $n + 1$ and fold the straight street into a circle with $n + 1$ places. Given a circular situation with $n + 1$ spaces we can get rid of space $n + 1$ and straighten the street to get a one-way street with n spaces.

6. Consider the round-a-bout case with n drivers and $n+1$ parking spaces. Each driver has a favorite space among the $n + 1$ spaces. Tell why every sequence of preferences is a successful parking sequence in this case. In view of this, how many "successful" parking sequences are there for the round-a-bout case?

7. In the round-a-bout case there is always a space left empty after all drivers have parked. Explain why if space $n + 1$ is empty then the sequence would also be a successful parking sequence for the straight street problem.

8. How many round-a-bout parking sequences leave space $n + 1$ empty? To answer this question, imagine any round-a-bout parking sequence and add 1 to every drivers favorite parking space number, where we take $(n + 1) + 1 = 1$ (clock arithmetic!) Comparing the two parking sequences, what can be said about the spaces left blank in each case?

9. Combine the results of problems 6, 7, 8 to find the number of round-a-bout parking sequences that leave space $n + 1$ empty.

10. How many parking sequences are there for a straight street with n parking spaces? Does your answer agree with the data gathered in the warm-up problem?

Where Can I Park?—Notes

Warm-up problem. You should find 1 sequence for $n = 1$, 3 for $n = 2$, and 16 for $n = 3$. Although there is not much data to go on here, you might observe in the $n = 3$ case that every permutation of a parking sequence is also a parking sequence.

1. There can be at most one person who prefers space 12. In addition, there can be at most two people who prefer spaces numbered 11 or higher. The collection of people who prefer 9 or 10 or 11 or 12 cannot be bigger than 4.

2. We also must have at least one person who prefers space 1. If not, everyone will drive past space 1; it will be "unparked" and we will be trying to squeeze n cars into $n-1$ spaces. By similar reasoning, the must be at least two people who prefer space 1 or space 2.

3. There are many ways to express the principles suggested in Problems 1 and 2. One way is this:

   > Given a successful parking sequence, if the parking preferences are ordered in increasing order $p_1 \leq p_2 \leq \cdots \leq p_n$, then it must be the case that $p_i \leq i$ for each i.

   To see how this works, consider the $n = 12$ case again and suppose, say, $p_9 > 9$. Then we also have $p_{10}, p_{11}, p_{12} > 9$. This means that four people (at least) prefer the last three places. Such a parking sequence cannot be successful.

5. The converse of the above principle also holds:

   > Suppose we are given a sequence of parking preferences. Rearrange the list of sequences to get a list $p_1 \leq p_2 \leq \cdots \leq p_n$ in nondecreasing order. If $p_i \leq i$ for each i, then the original sequence is a parking sequence.

   To see why this is true, again consider the case $n = 12$ and let $i = 9$ There are at least 9 drivers with parking preferences among the first nine places. The first 9 of these who arrive will take places 1–9. As for the other 3, if they prefer a space with a higher number than 9, they will by pass the first nine and will have an empty space among last three. For example, if we have a driver who prefers space 10, then $p_{10} = 10$ and $p_{11}, p_{12}, \geq 10$. Space 10 will never be taken by one of those who prefer spaces 1–9 because there are only 9 such drivers. The other two drivers must both prefer 10, both prefer 11, or prefer one each of 10, 11, 12. In all cases all cars can park. This is not a very rigorous argument, but the other cases ($p_{10} \leq 9$ also, etc.) can be argued in a similar way.

7. For the round-a-bout case, every parking sequence is successful. If a driver cannot park on his first time around he continues past (occupied) space $n + 1$ and parks in the first free space encountered on the second round. How many such sequences are there? Such a sequence has the form $\{s_1, s_2, \ldots, s_n\}$. Each of the n drivers can choose any of the $n + 1$ spaces. Thus there are $(n + 1)^n$ possible sequences and all are successful round-a-bout parking sequences.

8. Adding 1 to each drivers parking preference number just advances the empty space by 1. This means that if we examine all of the sequences, each space will be left empty as many times as any other space.

9. Because there are $n + 1$ spaces, we conclude that space $n + 1$ is empty for $\frac{1}{n+1}$ of the sequences. Thus the number of round-a-bout sequences that leave space $n + 1$ empty is

$$\frac{1}{n + 1}(n + 1)^n = (n + 1)^{n-1}.$$

Moreover, this is also the number of successful parking sequences for the straight street.