Lockers—an open and shut case—Teachers' Circle, Feb. 2009

Many of you are familiar with "the locker problem:"

Problem 1. At a large high school, there are 10000 lockers all on one wall of a long corridor. The lockers are numbered, in order, 1, 2, $3, \ldots, 10000$, and to start, each locker is closed. There are also 10000 students, also numbered 1, 2, $3, \ldots, 10000$. The students walk the length of the corridor, opening and closing lockers according to the following rules.

1. Student 1 opens every locker.
2. Student 2 closes every second locker.
3. Student 3 changes the state of every third locker, closing it if it is open, and opening it if it is closed.

$\vdots$

k. Student k changes the state of every k-th locker

$\vdots$

a. When all 10000 students have walked the corridor, which lockers end up open?

b. If the students go down in a different order, is the result changed?

c. What if student 3 is ill and had to skip her turn? What if she took a second turn when the teacher was not looking?

d. What if students 3 and 9 are ill? 3 and 10?

Problem 2. Suppose that we can send any students we like down the corridor. If, when we are done, we want only locker 1 open and all others closed, then which students should go? What if we want only locker 3 open?

Problem 3. Suppose that we want only the lockers with prime numbers open. Which students should be sent down the corridor?

Problem 4. Let L be any subset of $\{1, 2, \ldots, 10000\}$, the set of the first 10000 positive integers. Is there a set of students that you can send down the corridor so that when all of these students have gone, the set of open lockers is exactly those with a number in L?

Problem 5. Let S_1 and S_2 be two different groups of students. Each is sent down a row of lockers. Is it possible that the students from the two groups leave exactly the same lockers open?

—over—

Problem 6. A number is called square-free if it is not divisible by the square of any prime number (so the number 1 is square free.)

a. List the first 15 square-free numbers.
b. If we send all of the students with square free numbers down the corridor, which lockers will be open when the activity is done?

Problem 7. Suppose we want only locker 3 open. Which students should be sent down the corridor? What if we want only locker 9 open? Both lockers 3 and 9 and no other lockers?

Problem 8. Suppose we want only the lockers with prime numbers open. Which students should we send down the row of lockers?

Problem 9. Suppose that we send down the corridor exactly those students with perfect square numbers (e.g., 1, 4, 9, 16, ...) Which lockers are left open when this activity is concluded? What if we send only the students with numbers that are perfect cubes?