Fun with Folding and Pouring

Steve Dunbar and Anne Schmidt

October 10, 2013

Note: This lesson is based on Chapter 9, *Mathematics Galore!* by James Tanton, 2012, MAA, Washington DC.

Preliminary Folding Investigation

1. Take a strip of paper, fold it in half, and make a good crease at the midpoint position.

2. Open up the strip, and mark the crease "Midpoint".

3. With the strip unfolded, fold the *left* end over to meet the "Midpoint". Make a new crease halfway between the "Midpoint' and the left end of the strip by folding the left end of the paper.

4. Open up the strip again, and mark the new crease "Fold 1".

5. With the strip unfolded, make a new crease halfway between the "Fold 1" and the right end of the strip by folding the right end of the paper to meet the crease marked "Fold 1".

6. Open up the strip again, and mark the new crease "Fold 2".

7. Repeat, alternating left and right folds, with each fold made to the most recent crease mark. Mark each successive crease with $3, 4, \ldots$.

8. The sequence of crease marks seems to converge to two positions on the strip, what are they?

9. With a new strip of paper, repeat the experiment, except this time make the initial crease mark *anywhere* on the strip, not at the midpoint. The sequence of crease marks seems to converge to two positions on the strip, what are they this time?
   Answer: Should be 1/3 and 2/3.

Mathematical Analysis

Suppose the strip is one unit long, and the initial crease is at arbitrary position x ($0 < x < 1$ measured as a fraction of the length of the strip) as in the last step above.

Suppose the strip is one unit long, and the initial crease is at arbitrary position x ($0 < x < 1$ measured as a fraction of the length of the strip) as in the last step above.

1. Then a left fold to an arbitrary position x creates a new crease 1 at what position? (Express algebraically in terms of x.)

$$\frac{x}{2}$$

2. Now a right fold to a position x creates a new (even-numbered) crease at what position? (Express algebraically in terms of x.)

$$\frac{1}{2} + \frac{x}{2}$$

3. So with the second experiment, making the initial crease mark x *anywhere* on the strip, not at the midpoint, algebraically describe the position of the first 4 folds, left, right, left, right. (Hint: use compositions of functions.)

$$\frac{x}{2}, \frac{1}{2} + \frac{x}{4}, \frac{1}{4} + \frac{x}{8}, \frac{1}{2} + \frac{1/8}{+}\frac{x}{16}$$

Aside on base 2

In base ten arithmetic, the decimal $0.abcd\ldots$ represents

$$\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \frac{d}{10000} + \ldots.$$

Now think about base-2

1. In base two arithmetic, $0.abcd\ldots$ would represent what?
   Answer:
$$\frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \ldots$$

2. We can represent every real number x, $0 \leq x \leq 1$ in base two as $0.abcd\ldots$.

   (a) What is $\frac{3}{4}$ in base two?

   (b) What does $0.11111\ldots$ represent in base 2? (*Hint:* Remember geometric series!)

   (c) What is $\frac{1}{3}$ in base two?

3. If
$$x = \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \ldots$$
   so $x = 0.abcd\ldots$, then what is
   Answer:
$$2x = a + \frac{b}{2} + \frac{c}{4} + \frac{d}{8} + \ldots$$

4. Describe what multiplying by 2 does to the "decimal point". (Maybe we should call it the "binary point" or the "binimal point"!)

5. In base two, if:
$$x = \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \ldots$$
   so $x = 0.abcd\ldots$, then what is:
   Answer:
$$\frac{x}{2} = \frac{a}{4} + \frac{b}{8} + \frac{c}{16} + \frac{d}{32} + \ldots$$

6. Describe what dividing by 2 does to the "decimal point". (Maybe we should call it the "binary point" or the so $2x = 0.0abcd\ldots$, so multiplying by 2 shifts the "binary point".

Back to Paper Folding

1. If the initial crease is at $x = 0.abcd\ldots$ (in binary) then a left fold puts a new crease where?
   Answer:
   $$\frac{x}{2} = 0.0abcd\ldots\ldots$$
   so it inserts a 0 into the first space in the binary expansion of x and pushes the rest to higher places.

2. A right fold puts a crease at
   $$\frac{1}{2} + \frac{x}{2}$$
   Describe the action of this in binary notation. (*Hint:* recall that $\frac{1}{2} = 0.1$.)
   Answer: so it inserts a 1 into the first space in the binary expansion of x and pushes the rest to higher places.

3. Thus if we make four left and right folds, where will be the latest crease?
   Answer: at $0.0101abcd\ldots$.

4. If we make ten left and right folds, where will be the latest crease?
   Answer: crease at $0.0101010101abcd\ldots$.

5. With more and more folds, what values will we be approaching?
   Answer: we will approach the values $0.01010101010\ldots$ which is $\frac{1}{3}$ and $0.101010101010\ldots$ which is $2/3$respectively. The folds converge to these two numbers.

Comment on Pouring Buckets

Paper folding is equivalent to a water transfer problem: On gallon of water is transferred between two containers labeled A and B. Half the contents of A are poured into B (changing contents of A by $x \mapsto \frac{x}{2}$ and then half the contents of B and poured back into A (changing contents of A by $x \mapsto x + \frac{1-x}{2} = \frac{1}{2} + \frac{x}{2}$. The process of alternately pouring half form A to B and then B to A is repeated. What happens in the long run?

Question: Instead of transferring half the contents, suppose we transfer $\frac{3}{4}$ of A to B and then $\frac{1}{2}$ of B back to A. What happens to the amounts in each bucket over the long run?

If the original pattern of "half" pouring between A and B is represented as $A \to B, B \to A, A \to B \dots$ then explain what would be the results of the pattern $A \to B, A \to B, B \to A, A \to B, A \to B, B \to A \dots$.

What happens if you try other patterns of pouring?

Student Research Experiment and Project

Consider a discrete (and less messy!) version of the pouring problem:

1. Suppose that we have 14 marbles in one cup (or can or bucket) labeled A and 18 marbles in a second cup (or can or bucket) labeled B.

2. Pour (or count out) half the contents of cup A into cup B ...

3. ... and then half the contents of cup B back into A keeping the extra odd marble in B.

4. Make the (arbitrary) rule *Cup B always keeps or is given any extra odd marble from the "halving".* Repeatedly pouring half (or just over half in the odd case) of the contests of A into B and then half (or just under half in the odd case) of the contents of B into A, what happens?

Questions for Research

1. Does every initial distribution of 32 marbles lead to the same outcome as for the 14-18 initial distribution?

2. Start with a different initial number of marbles and repeat the experiment above for different distributions. *Hint:* Keep the number of marbles small, say try the experiment with 8 marbles in various distributions, and again with 9 marbles in various distributions.

3. What is characteristic about the outcomes? What are the common characteristics and different characteristics among the numbers 8, 32 and 9.

4. Prove that if the number of marbles with which we start is a power of two, then every initial distribution enters into the same oscillation.

5. Does it make a difference if we change the arbitrary rule about how to handle the case of an odd number of marbles in either can?

6. What if we change the proportion transferred from something different from a half?

7. Is it possible to find a game (total number of marbles, initial distribution of marbles, rule about handling odd numbers of marbles, etc.) that enters into a cycle with a period different from 2?