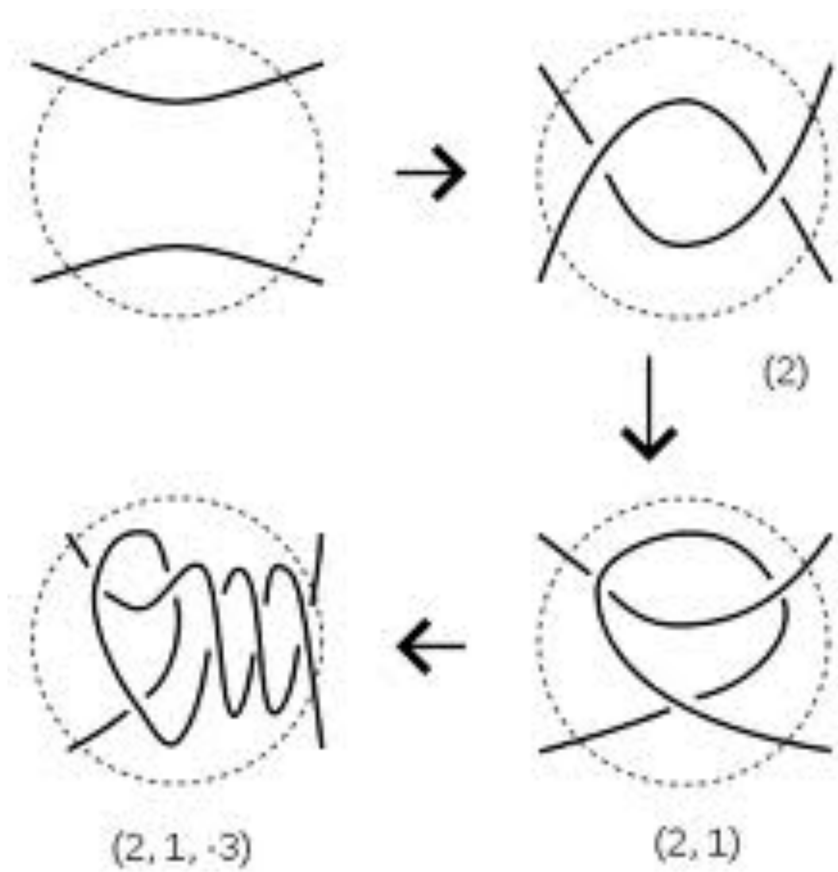
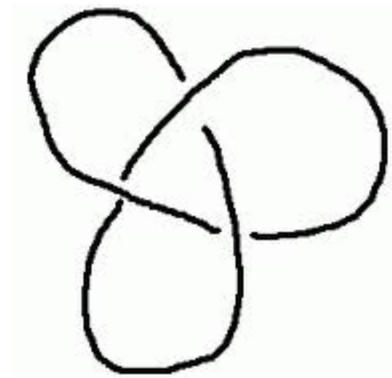
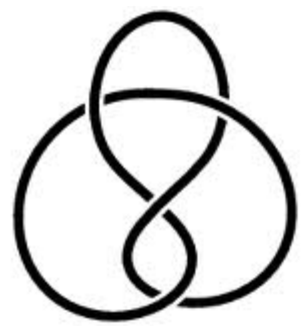
Conway's Rational Tangles

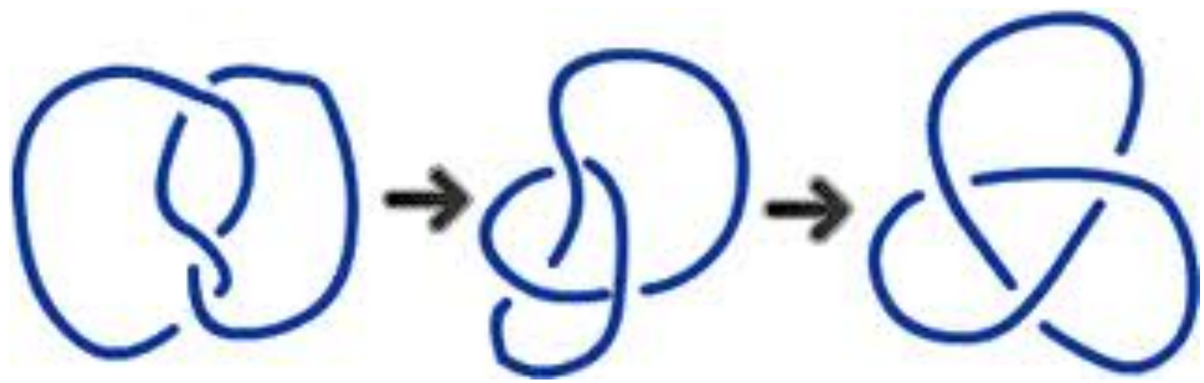


Knots

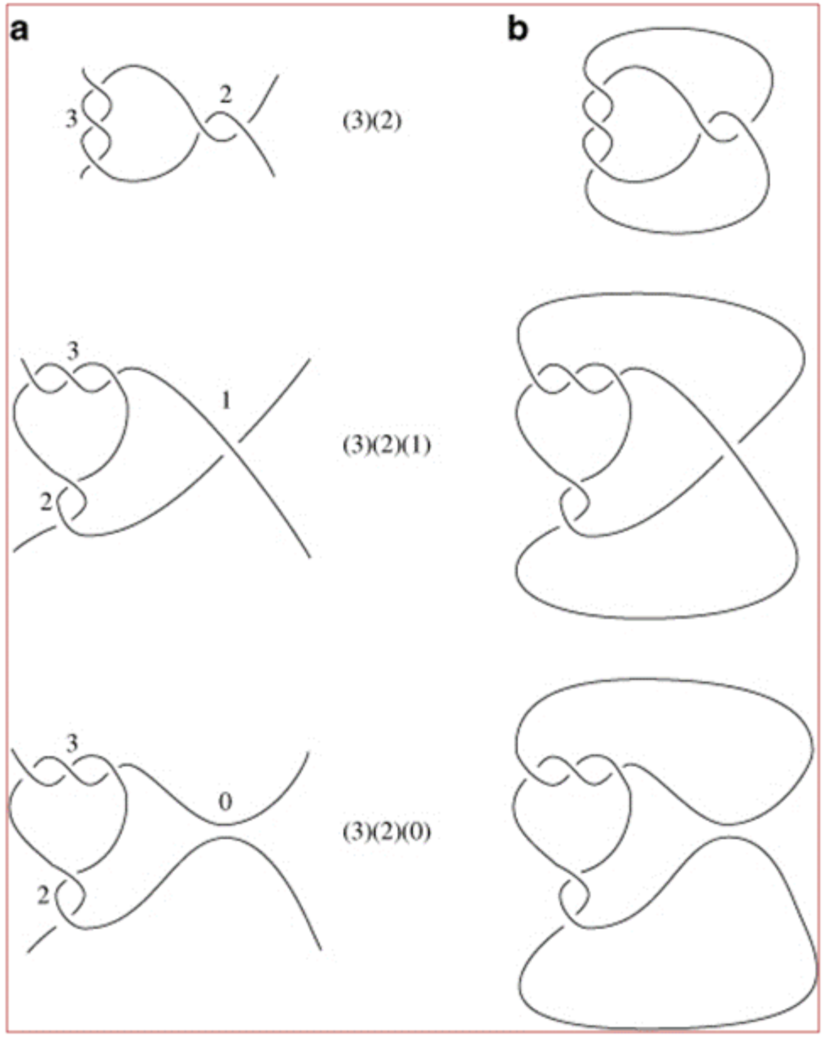





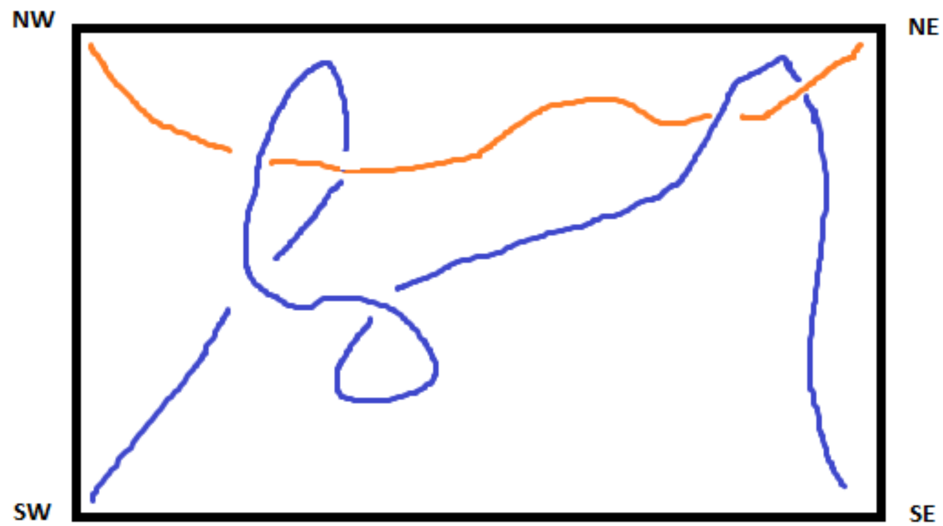
Equivalent Knots

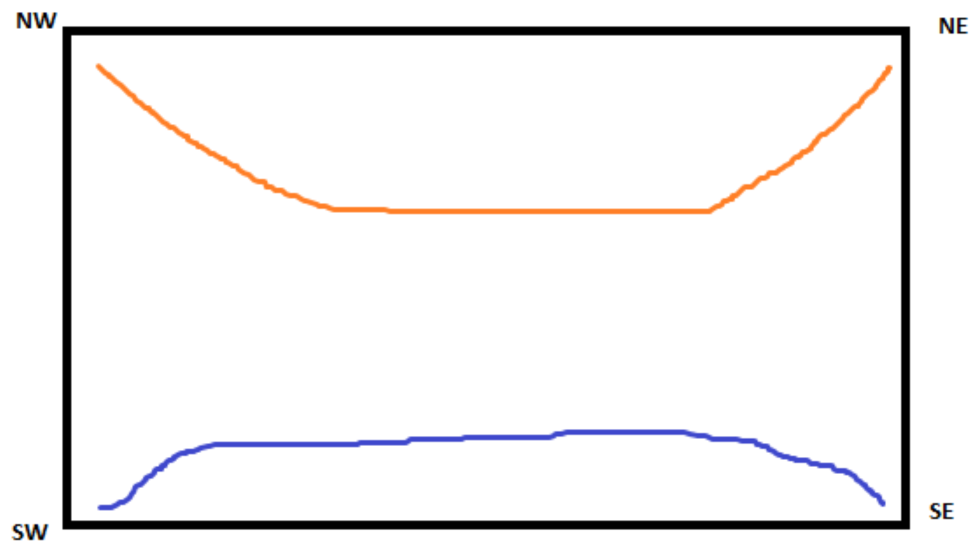


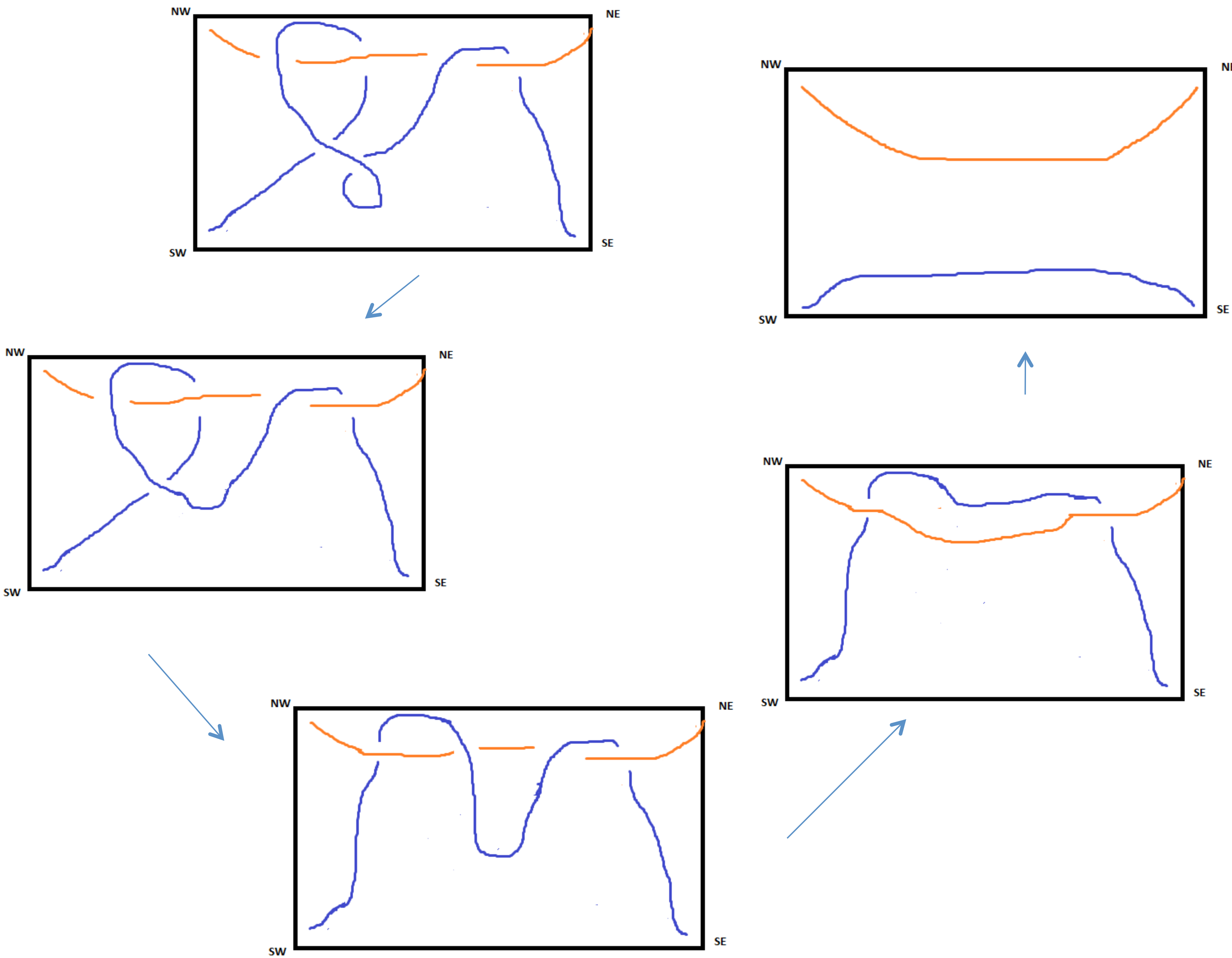
Knots and Tangles



Are these tangles equivalent?







NW
NE
SW
SE
NW
NE
SW
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NW
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SW
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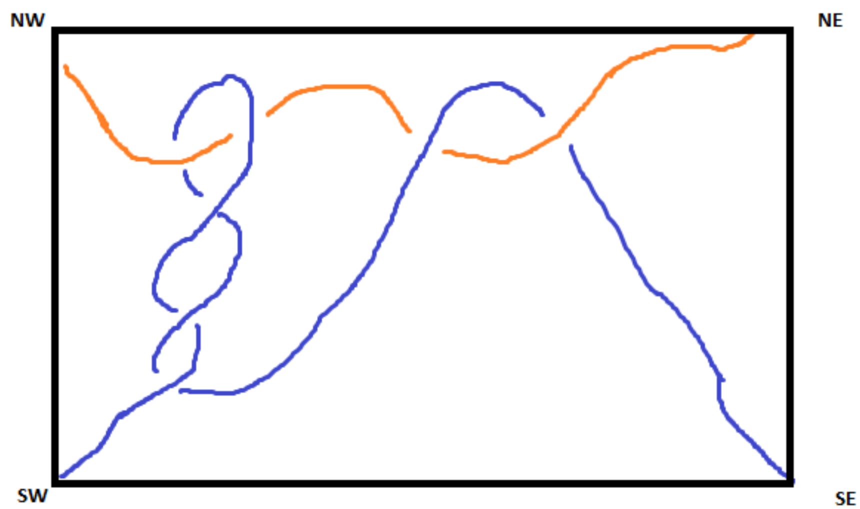
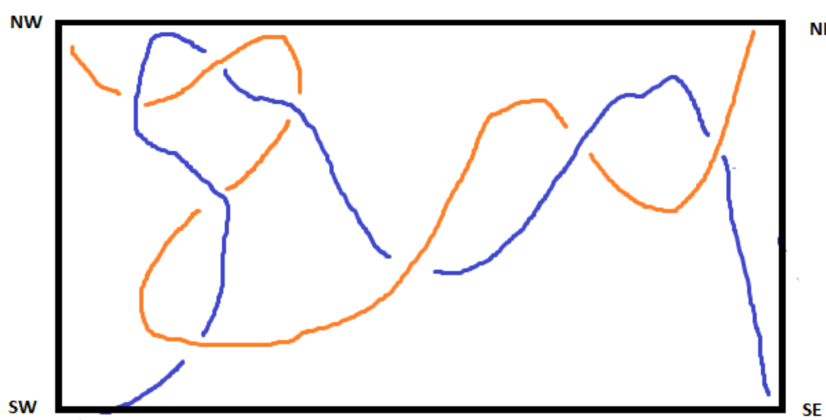
NW
NE
SW
SE

Are these tangles equivalent?



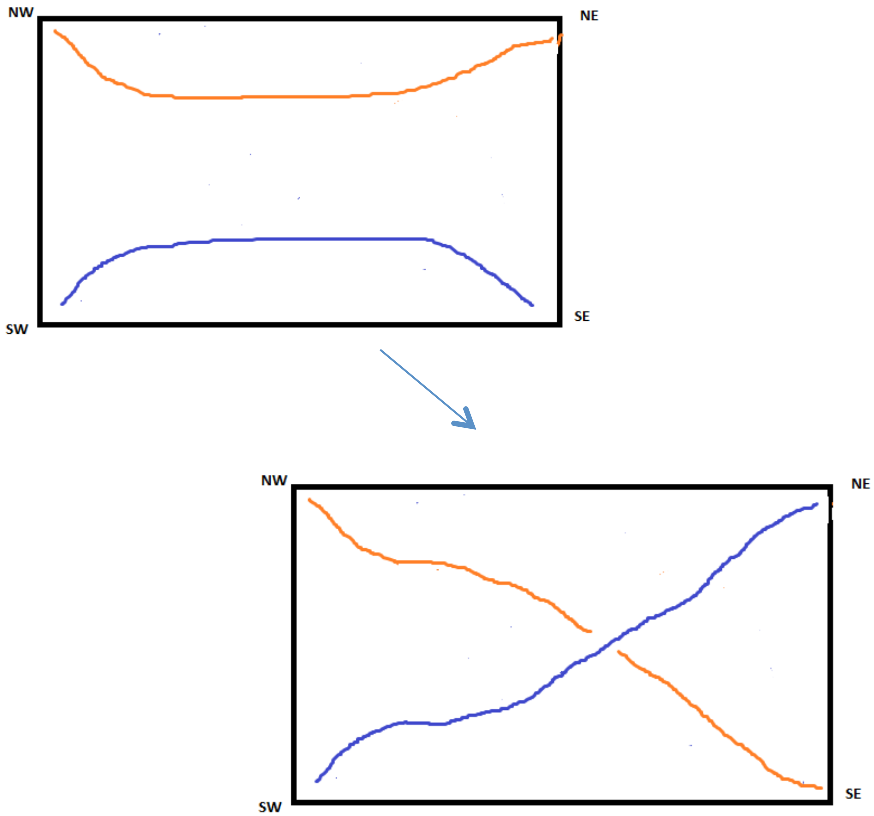


What I like doing is taking something that other people thought was complicated and difficult to understand, and finding a simple idea, so that any fool - and, in this case, you - can understand the complicated thing.

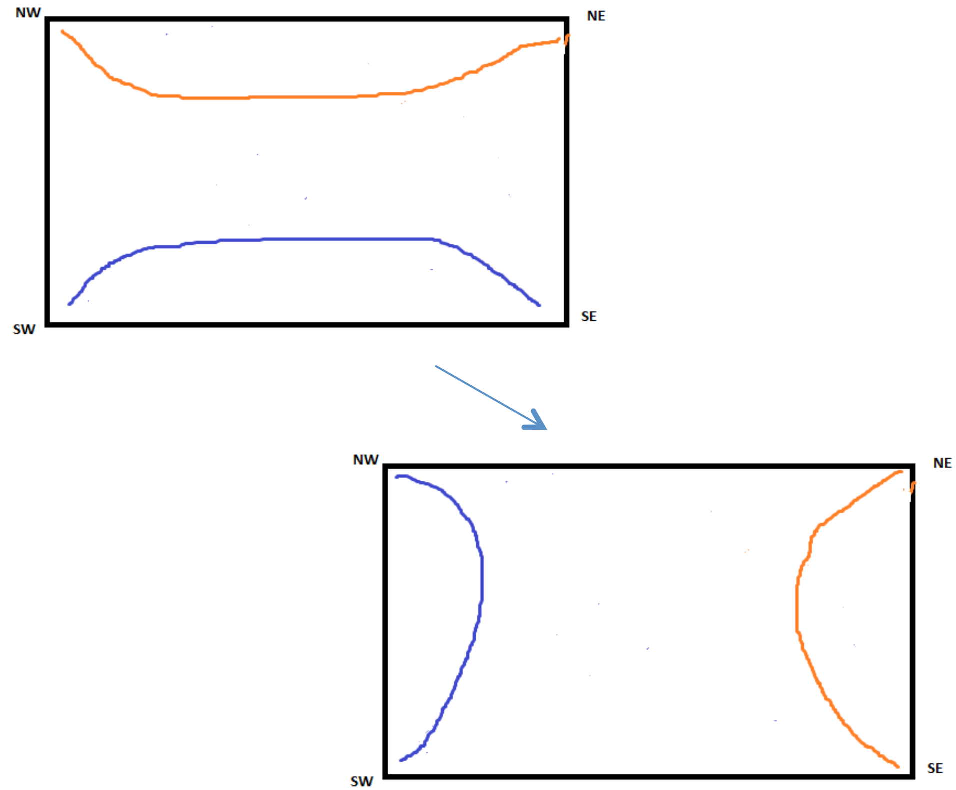
-John Conway

Conway’s Dance Steps

The Twist (“twist ‘em up”)

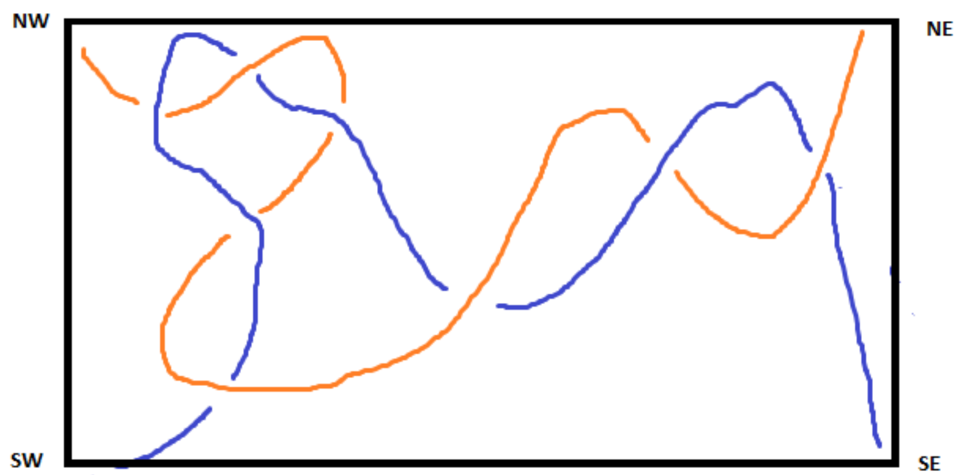


The Rotate (“turn ‘em around”)



Conway's Rational Tangle Game

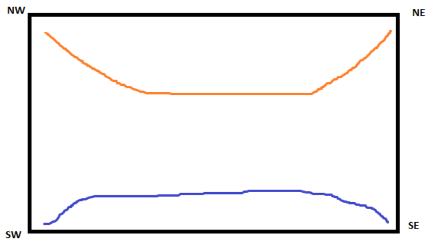
Can you untangle any tangle using only the Twist and the Rotate?



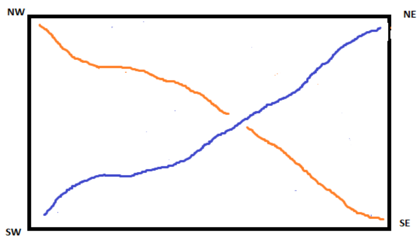
The Tangle Number

Conway's simple idea: Associate a number to each tangle and describe how each dance step affects the tangle number.

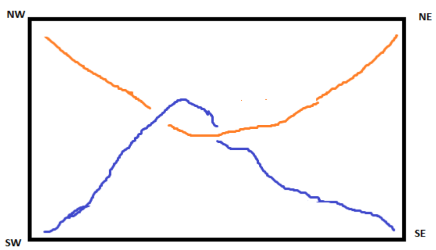
Twisting



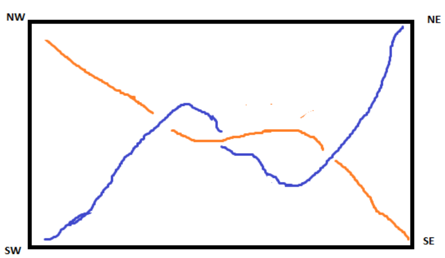
The 0 tangle



Tangle number = 1



Tangle number =2



Tangle number = 3

How do the dance steps change the tangle number?

The Twist

$$x \xrightarrow{\ \mathbf{T}\ } x+1$$

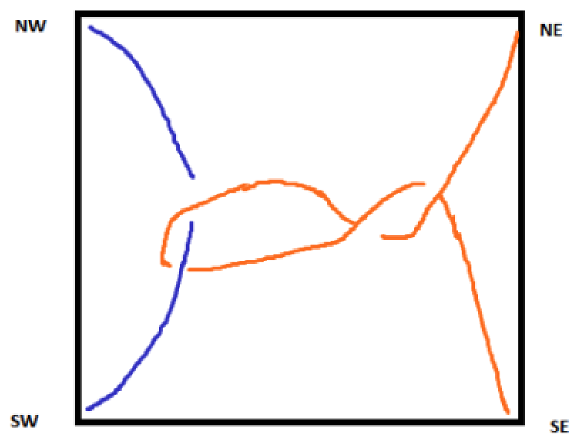
The Rotate

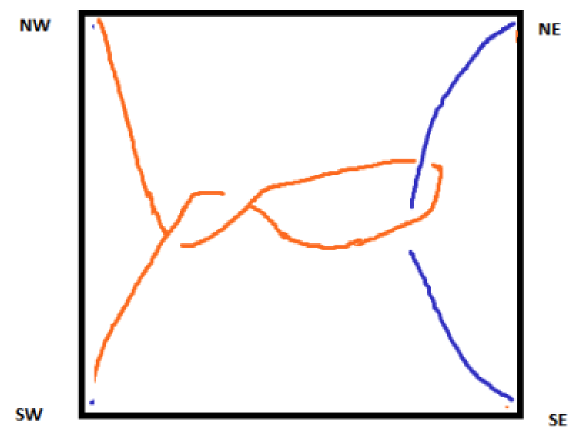
$$x \xrightarrow{\ \mathbf{R}\ } ????$$

Investigating the Rotate

Here are some sequences of moves that will help us figure out what the Rotate does to the tangle number:

1. Start with the 0 tangle and rotate twice (RR). What tangle do you get?
2. Start with TT and rotate twice. What tangle do you get? What do you conjecture about rotating twice?
3. Make the tangle TTRTTT, and have your neighbor make TTRTTTRR. Are the two tangles equivalent?





More about the Rotate

1. Start with the 0 tangle and twist, then rotate, then twist (TRT). What do you get? So rotating changes tangle number 1 to what tangle number?

$$0 \xrightarrow{T} 1 \xrightarrow{R} ? \xrightarrow{T} ?+1$$

2. Start with 0 then do TTRTT. What does this tell you?

3. Start with 0, then do *RT*. What do you get?

$$0 \xrightarrow{R} ? \xrightarrow{T} ?+1$$

Effect of the dance steps on the tangle number

T

$$x \longrightarrow x+1$$

R

$$x \longrightarrow -\frac{1}{x}$$

Find the tangle numbers:

- TTTRT
- TTRTTTRT
- TTTRTTTTR

Solutions

$$0 \xrightarrow{T} 1 \xrightarrow{T} 2 \xrightarrow{T} 3 \xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3}$$

$$0 \xrightarrow{T} 1 \xrightarrow{T} 2 \xrightarrow{R} -\frac{1}{2} \xrightarrow{T} \frac{1}{2} \xrightarrow{T} \frac{3}{2} \xrightarrow{T} \frac{5}{2} \xrightarrow{R} -\frac{2}{5} \xrightarrow{T} \frac{3}{5}$$

$$0 \xrightarrow{T} 1 \xrightarrow{T} 2 \xrightarrow{T} 3 \xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{T} \frac{5}{3} \xrightarrow{T} \frac{8}{3} \xrightarrow{T} \frac{11}{3} \xrightarrow{R} -\frac{3}{11}$$

Untangling the tangles

Untangle TTTRT

Untangle TTRTTTRT

Untangle TTTRTTTTR

Do you have a strategy that always works? Can every rational tangle be untangled?

Tangling the tangle

Starting from the 0 tangle, can we get to any positive or negative rational number?

How would you get to $-n$ for any positive integer n?

How would you get to $\frac{1}{n}$ for any positive integer n?

How would you get to $\frac{n}{n+1}$ for any positive integer n?

How would you get to $\frac{n}{n-1}$ for any positive integer n?

- Each tangle is associated with a continued fraction.

Example: TTRTTTRTTTT is represented by

$$4+\frac{1}{-3+\frac{1}{2}}$$

- Conway showed that two rational tangles with the same tangle number are equivalent.

- Find the continued fraction representations of TTRTRTRTRTTRTTTT

  and TTRTTRTTTT

  Show that the tangles are equivalent.

  Show that inserting TRTRTR into any sequence of twists and rotates has no effect on the tangle.

  Are there other sequences that are trivial in this way?