MTCircular

Summer/Autumn 2016

Quilt While You're Ahead

Investigating Quilt Block Symmetries

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Math Teachers' Circle π

Connecting a Growing Network

Dear MTC Network,

Our network has been growing in exciting ways over the past few months. In February, we were selected as a new partner in 100Kin10 (**www.100kin10.org**), a national network focused on training and retaining 100,000 excellent STEM teachers by 2021. Being a 100Kin10 partner provides opportunities not only to make new connections and learn from the experience of others working in STEM education, but also to share our work outside our typical sphere. For example, I highly recommend checking out a recent webinar for Educator Innovator, a program of fellow 100Kin10 partner the National Writing Project, in which Chris Bolognese, Fawn Nguyen, Paul Zeitz, and Joshua Zucker discuss “What Makes a Good Problem?” (**www.educatorinnovator.org/webinars/what-makes-a-good-problem**).

It was my great pleasure to be part of the launch of the new Montana MTC Network at a workshop in April, along with Angie Hodge, Bob Klein, and a group of dedicated Montana math professionals led by David Patterson, Fred Peck, Matt Roscoe, and Ke Wu of the University of Montana. In June, Nathan Borchelt and Sloan Despeaux organized a dual-purpose workshop that inducted new teachers into the Smoky Mountain MTC and also launched new MTCs in North Carolina, South Carolina, and Virginia, with the help of mentors Mark Brown and Joshua Zucker. Our mentoring program for new MTCs is also in full swing, with new leaders visiting nearby MTCs and experienced leaders visiting new MTCs across the country.

You’ll notice that this issue of the *MTCircular* contains more math than ever, with write-ups of four intriguing new sessions. I hope many of you try them out with your MTCs this year!

Happy problem solving!

Brianna Donaldson

Brianna Donaldson, Director of Special Projects

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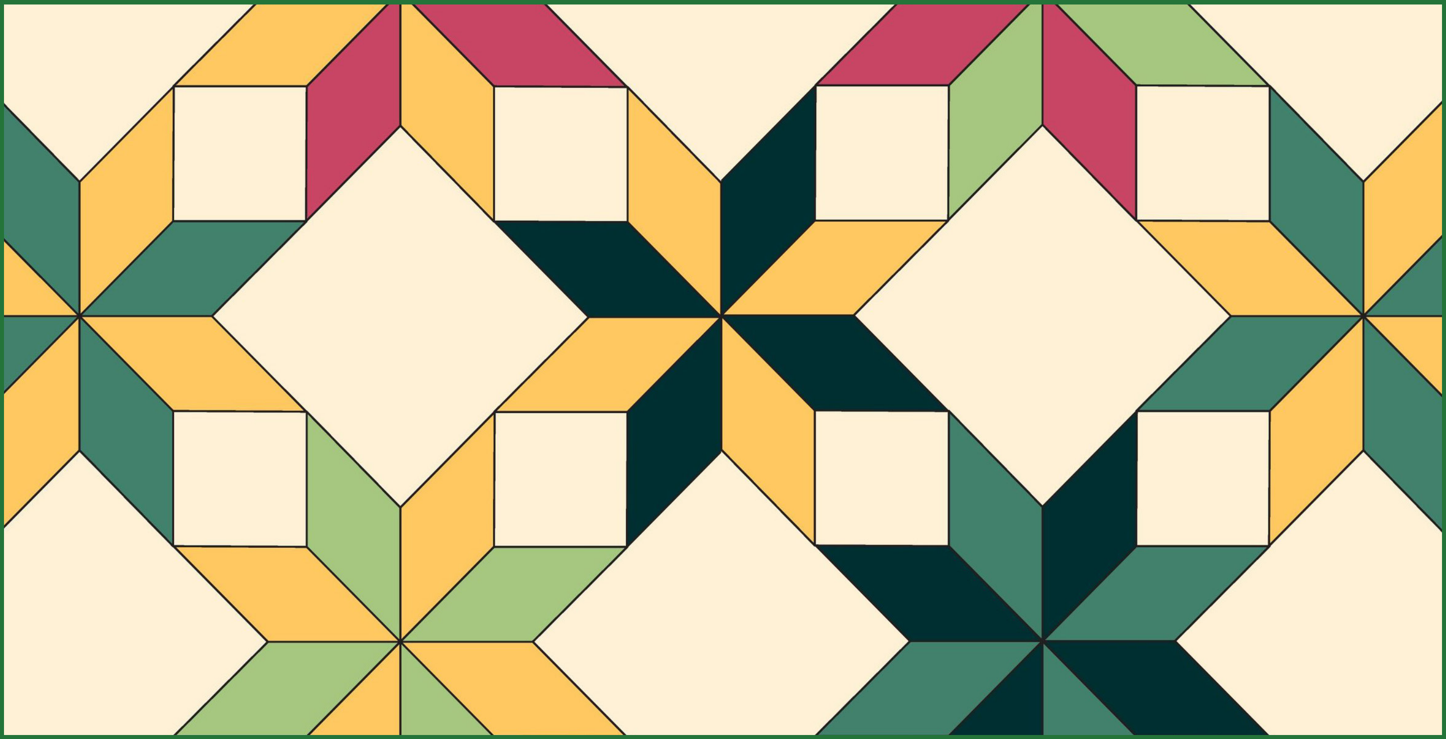
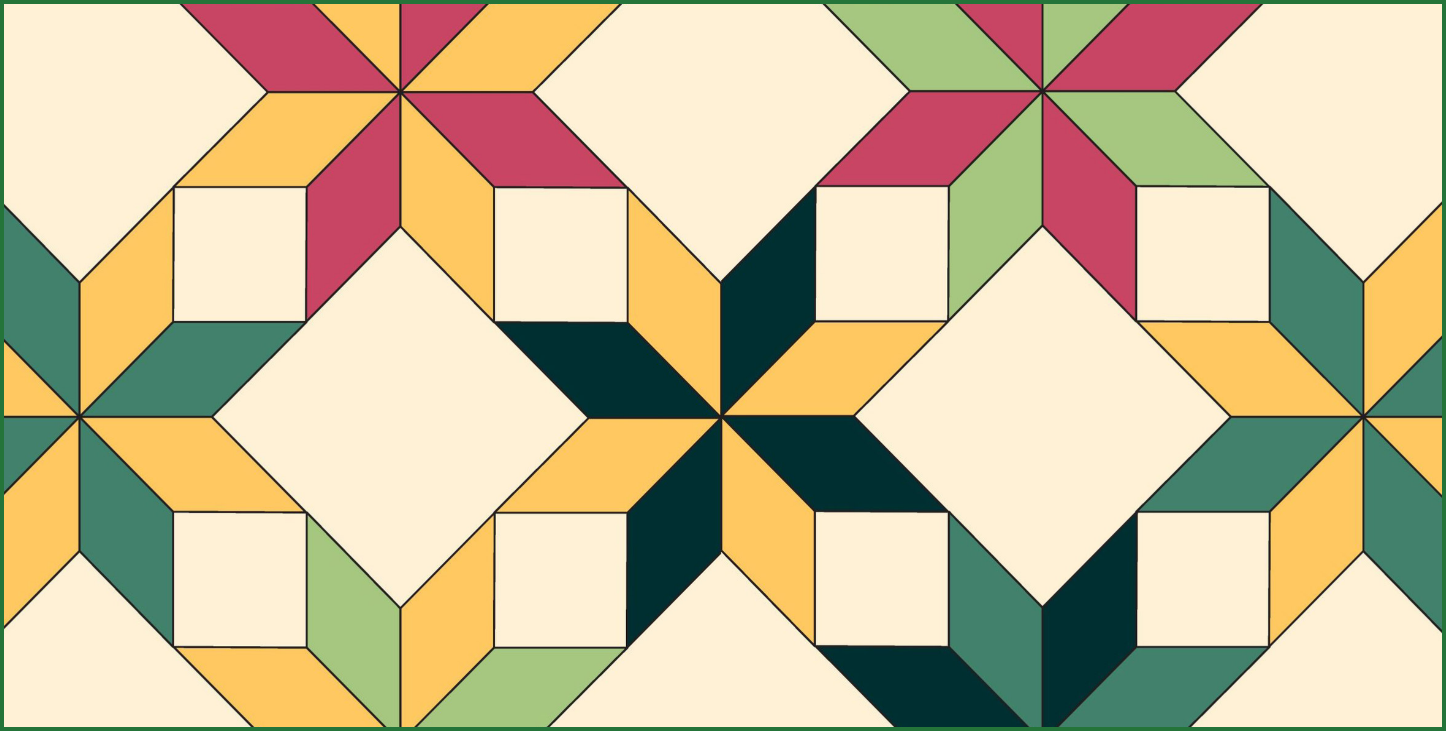


Quilt While You're Ahead

Investigating Quilt Block Symmetries

by Matt Roscoe



"A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas." (G. H. Hardy)

"...in mathematics, the primary subject-matter is not the mathematical objects but rather the structures in which they are arranged." (M. Resnik)

Quilts are a familiar set of cultural artifacts for many people. Prized for their unique patterns and labor-intensive production, quilts also happen to be beautifully mathematical.

A primary unit of quilt making is the quilt block, a square grid filled with patches of fabric according to certain rules. Quilters commonly refer to quilt blocks as *9-square* or *16-square* because of the underlying grid. In this exploration, we will consider the 16-square. The reader is invited to extend (or should I say reduce?) the exploration to the 9-square.

Introduction: Build a Quilt Block

To launch the investigation, I ask participants to imagine that they are planning a 16-square quilt block. Now, suppose that we restrict each quilt block to a 4-by-4 grid of squares. Each square can be filled in one of six ways, as shown in Figure 1. One possible quilt block configuration is shown in Figure 2.

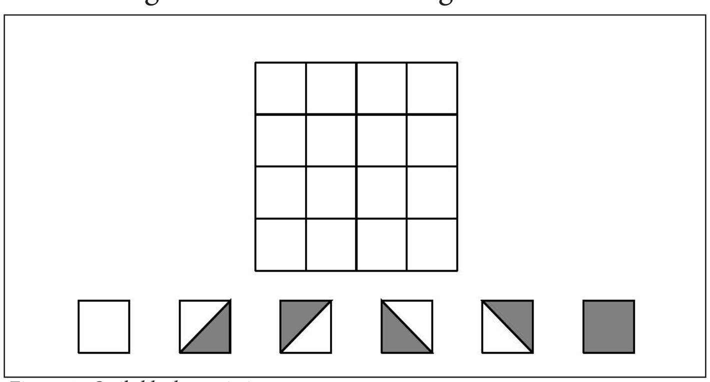


Figure 1. Quilt block restrictions

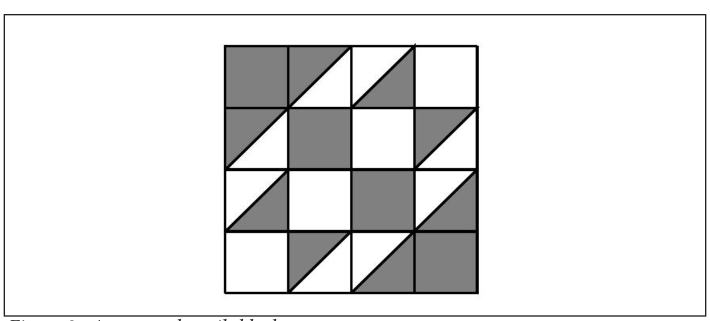


Figure 2. An example quilt block

Stock Photo: Natalya Belinskaya, 123rf.com.

Once participants understand the rules by which quilt blocks can be constructed, I ask them to color a quilt block using a blank 16-square grid.

Compare Your Blocks, Find Your Partners

Once everyone has created a quilt block or two, I ask participants the question, "From a mathematical perspective, how might your quilt block be similar to or different from the one your neighbor has constructed?" Inevitably, this starts up a conversation about fractions, decimals, and percents. After we agree that a classification scheme could be imposed using this framework, I try to steer participants to something deeper by asking, "Why do humans find quilts beautiful?" Usually, this is enough to get participants thinking about symmetry. I ask, "What symmetries do your quilts possess? Do all quilts possess the same symmetries?" From here, I ask participants to circulate around the room, compare quilt blocks, and find a "partner" whose quilt block possesses exactly the same symmetries.

Developing a Taxonomy

Once participants have become familiar with the idea that all quilt blocks are not necessarily the same in terms of the symmetries they have, I bring the group back together and ask, "What sorts of symmetries can a quilt block possess?" Here, I hope to elicit the four possible line symmetries and the three possible turn symmetries. These symmetries, together with the identity symmetry, are displayed in Figure 3 below.

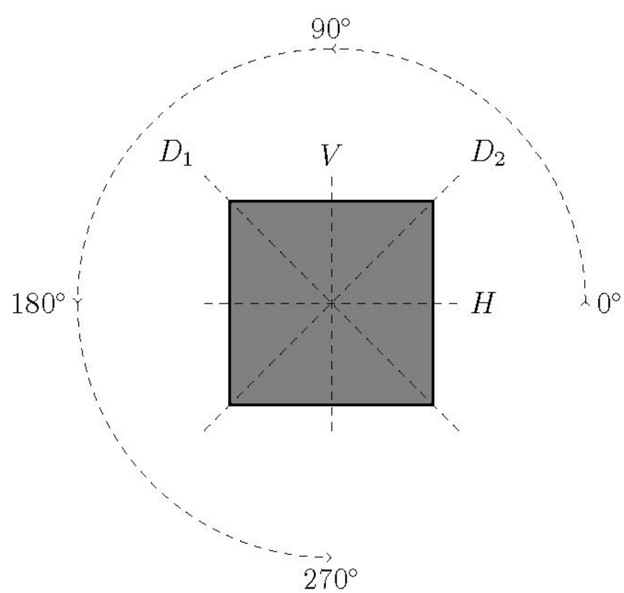


Figure 3. Available symmetries of the square

I encourage the participants to develop a taxonomy of symmetry so that we can compare the blocks according to the symmetries present/absent. Using the example in Figure 2, I ask, "What would we call this if we were to classify it according to its symmetries?" Usually, someone will say something like, "Diagonal, Diagonal, and 180." I argue that it might be easier to attach an algebraic representation such as *D-D-180*. This leads to a classification scheme that might look something like:

- *90*: 90-degree rotational symmetry
- *180*: 180-degree rotational symmetry
- *270*: 270-degree rotational symmetry
- *H*: reflective symmetry in a horizontal line; interchangeable with *V*
- *V*: reflective symmetry in a vertical line; interchangeable with *H*
- *D*: reflective symmetry in a diagonal line; two possible

Using this scheme, I ask participants to identify the different "types" of quilts they had constructed, an activity that leads naturally to the question, "Are there more?"

Are There More?

At this point, I hand out a set of 28 quilt blocks for further investigation. Organized into small groups, participants are encouraged to use the classification scheme to "type" each quilt according to the symmetries present. These 28 quilt blocks are displayed in Figure 4. The reader is encouraged to attempt this classification task before reading on.

A tool that may be useful in completing the activity is the Mira, a small plastic device that helps with the concept of reflective symmetry. The sorting activity takes time and invariably leads to conflict. Encourage groups to settle conflicts by presenting their reasoning or by critiquing the reasoning of others. Groups should come to the consensus that there are seven different symmetry classes, each containing four distinct quilt blocks:

- *D*
- *H-V-D-D-90-180-270* (also called "*ALL*")
- *D-D-180*
- *H*
- *H-V-180*
- *180*
- *90-180-270*

In Figure 4, members of these classes are found in the seven rows. This sorting activity usually adds several new

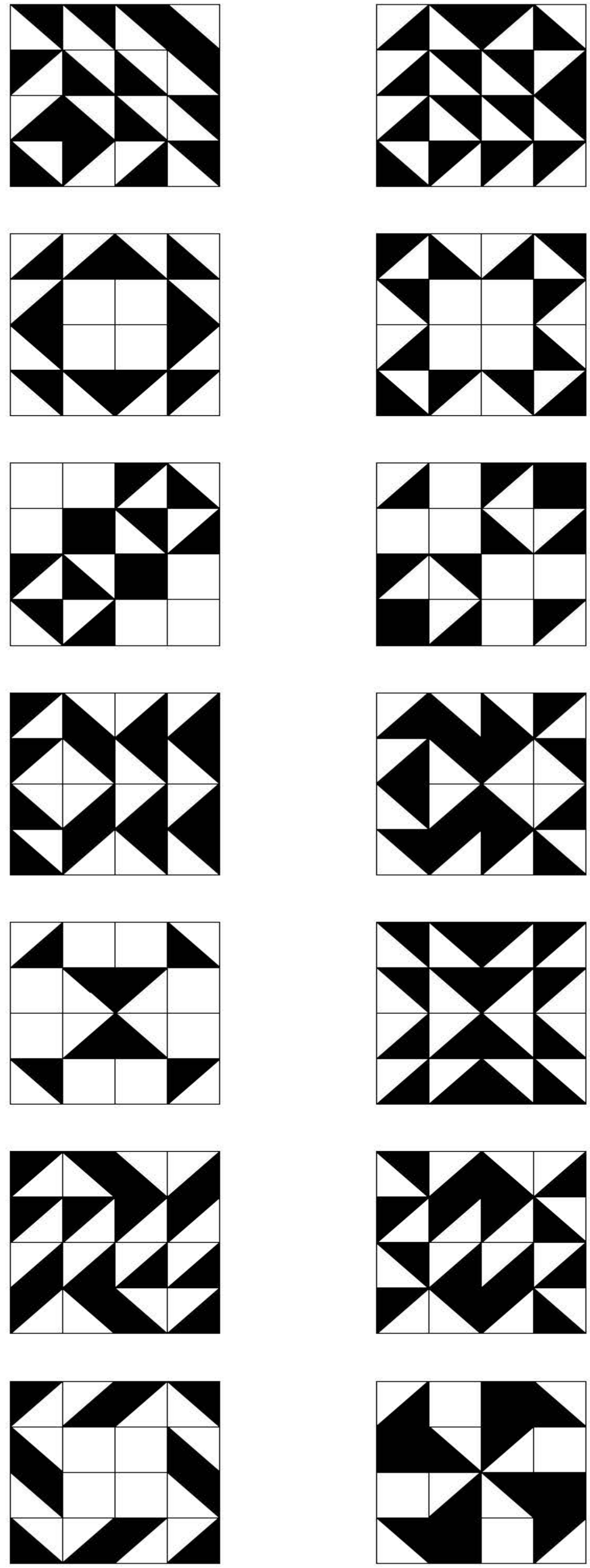
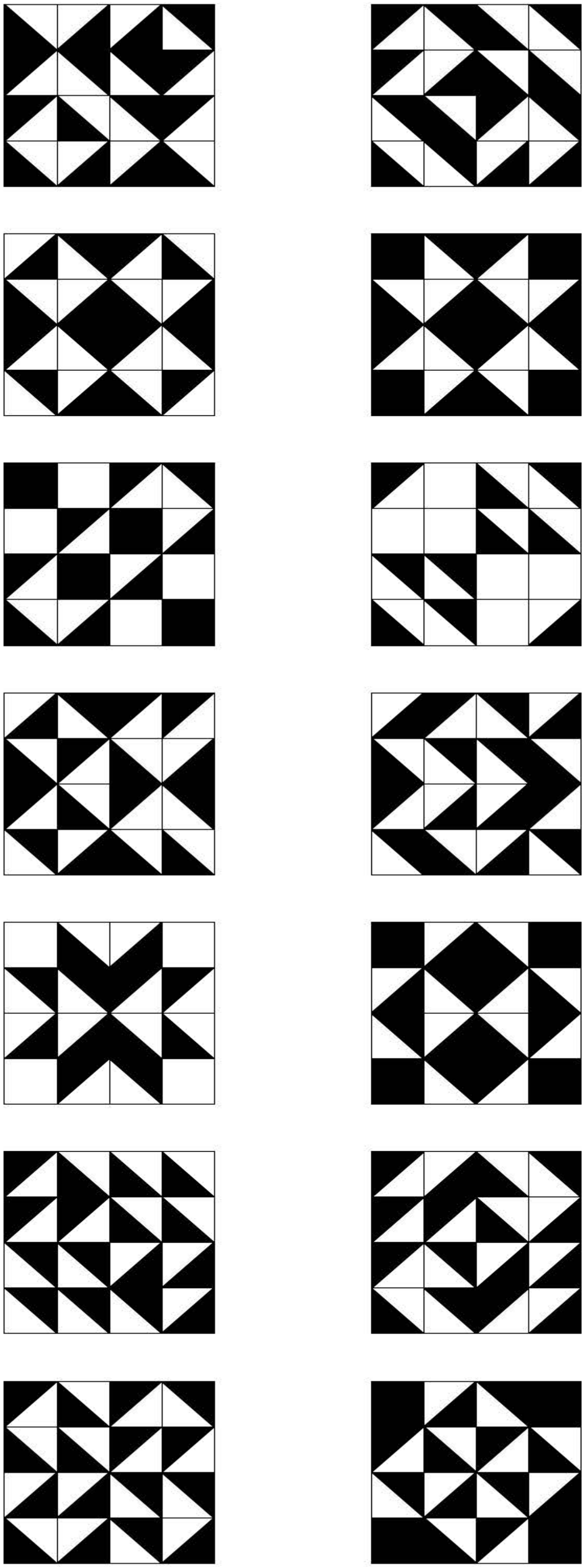


Figure 4. Twenty-eight quilts to sort by symmetry



classes to the list that had been "discovered" through construction. But the question remains: Are there more?

Are There More? Part II

To investigate this question further, I encourage participants to "imagine" other possible symmetry groups. For example, "Could a *D-D-90* quilt exist?" Usually participants come to the conclusion that if a quilt block has 90-degree rotational symmetry, then it must have 180-degree rotational symmetry—so, no such group exists. This naturally leads to the idea that a quilt block's symmetries can be composed to produce other symmetries—a 90-degree rotation composed with a 90-degree rotation is a 180-degree rotation. This sort of investigation can lead to many interesting results. Past participants have discovered that a quilt possessing any two line symmetries automatically inherits a rotational symmetry (disallowing types such *H-V* or *H-V-D* or *D-D*). Participants are also quick to note that any quilt with 90-turn symmetry inherits all turn symmetries (disallowing types such as *180-270* or *90-270* or *H-V-90*). Proceeding via elimination, we can conclude that there are only seven quilt block "types" that possess at least one symmetry. Each of these types is found in the list above.

Conclusion

A nice way to conclude the activity is to look at some actual quilts through the lens of symmetry groups. If you have no quilts on hand, pictures of quilts are easily found on the internet. Questions for further study include:

- How many quilts can be constructed according to the quilt block restrictions in Figure 1? (Here you might want to look for an upper bound first.)
- How many quilts that possess all possible symmetries can be constructed according to the quilt block restrictions in Figure 1? (Consider black and white "negative" images for an even more interesting result.)
- How many quilts that possess all the rotational symmetries (and no others) can be constructed according to the quilt block restrictions in Figure 1? (You might want to answer the previous question first!)
- Is the number of *D-D-180* quilt blocks the same as the number of *H-V-180* quilt blocks? (The answer might surprise you.)

For resources related to this article, please visit **www.mathteacherscircle.org/newsletter**.

Winning the Lottery

An Expected Value Mystery

by Michelle Manes

Many of our sessions at the Math Teachers' Circle of Hawai'i (MaTCH) are focused on giving teachers time to explore and problem solve, and the facilitators don't spend too much time "teaching" or explaining any particular mathematical content. Sometimes, though, we find an irresistible piece of mathematics that we want to share at MaTCH, and these sessions run a bit differently:

- Introductory activity to inspire / elicit a particular mathematical idea
- Teaching about that idea (usually a 10-15 minute PowerPoint or whiteboard talk followed by Q&A time)
- Follow-up activity that either extends the mathematical idea or ties it to other topics

The Inspiration

Here I'll describe a recent MaTCH session with this format, inspired by my friend Jordan Ellenberg's wonderful book *How Not to Be Wrong: The Power of Mathematical Thinking*. Ellenberg tells the story of a lottery scam that wasn't actually a scam:

> *In 2005, an MIT student was working on a senior project examining the expected value for lottery tickets in various states, and he noticed something incredible. On certain special weeks, the expected value of a \$2 Cash Winfall ticket in Massachusetts was about \$5.53.*
>
> *(In most weeks, the expected value for a ticket in this lottery, like in all lotteries, was much lower than the ticket price: 80¢ for a \$2.00 ticket. So those special weeks were really, really special.)*
>
> *Of course, that doesn't mean that every \$2 ticket would pay out \$5.53. As Ellenberg says, "expected value is not the value you expect." What it does mean is that if you buy lots and lots of tickets — thousands of them — then you have a good chance of almost tripling your money.*
>
> *After checking and re-checking and re-re-checking his work, this student got together with several of his friends and did just that. They bought thousands of lottery tickets each time the lottery had this special, high expected value week. And they won millions of dollars between 2005 and the last Cash Winfall drawing in 2012.*
>
> *Two other "cartels" noticed the same opportunity in the Massachusetts lottery and also started playing big on those special weeks. But the MIT cartel was different: Instead of using "Quick Pick" machines to choose numbers randomly, the MIT students undertook the tedious job of filling out their lottery tickets by hand.*

Ellenberg asks the obvious question: Why would anyone do that? He offers two explanations: First, birthday problem-like reasoning says that using the Quick Pick machine thousands of times will surely result in lots of duplicate tickets, something the cartels want to avoid. His second explanation has to do with guaranteeing a minimum return on the bet, and it ties to even more interesting mathematics: finite projective geometries.

Introductory Activity

We started the session by playing a mini lottery game. Each table had a banker to collect money for ticket purchases before each round and to pay out winnings after each draw. Each player started with \$10 in Monopoly money. The rules:

- We used standard playing cards numbered 1–7. A "ticket" was a choice of three numbers from 1–7, no repeats.
- Before each round, participants bought their tickets for \$1 each. They decided which tickets to buy and how many.
- Each round, the MC shuffled the mini-deck, drew 3 cards, and placed them face-up on the document camera.

The payout scheme:

- Match all 3: Win \$10
- Match 2 of 3: Win \$3
- Match 0 or 1 of 3: No prize

Everyone (including me) played ten rounds. If you lost all of your money, you were out and couldn't play any more. The MC kept track of the winning draws on a whiteboard, so that we could reference them later.

After all ten rounds, I asked everyone to stand up. Then I had them sit down if they:

- Were out of money,
- Had less than \$5 left,
- Had less than \$10 left,
- Had less than \$20 left,
- Had less than \$30 left.

When only a couple of people were left standing, we compared our winnings. I did better than all but one player by playing the same seven tickets each round:

1-2-3, 1-4-7, 1-5-6, 2-5-7, 2-4-6, 3-4-5, 3-6-7.

I put my seven tickets on the document camera and asked everyone to figure out what I won or lost in each round. They were surprised to find that not only did I win money overall, I never lost money. In each round, either I won the jackpot (up \$3 after buying seven tickets and winning \$10), or I won exactly three of the "match 2-out-of-3" prizes (up \$2 after buying seven tickets and winning \$9).

Some other players might have had bigger individual rounds (for example winning the jackpot and one or more 2-out-of-3 prizes). But no one consistently came out ahead in every single round, so they wanted to know how I had done it. I promised that we would come back to that question before the end of the session.

Mathematics: Expected Value

At this point, we took about 15 minutes to talk about the odds of winning a lottery and how they are calculated. The basic rules of probability tell us that

P(event) = (# ways that event can happen) / (total # of possible outcomes).

I asked them to calculate the probability of a single ticket hitting the jackpot in our 7-ball lottery. Some of the secondary teachers used combinatorics to calculate 7-choose-3. But the numbers are small enough that working systematically, it can be seen that:

P(jackpot) = 1/35.

The probability of a single ticket winning the 2-out-of-3 prize is harder to calculate, since you have to consider all of the possible pairs in the set of three cards (there are 3 pairs), and you have to remember that winning the 2-out-of-3 prize means that you didn't win the jackpot. Teachers worked for a while on this question, until someone was able to convince the group that

P(2-out-of-3) = 12/35.

We then defined the mathematical term "expected value" (EV), which is really more like a weighted average value:

EV = (Probability of Event #1)(Value of Event #1) + (Probability of Event #2)(Value of Event #2) + … + (Probability of Event #n)(Value of Event #n).

In our lottery, there are only two events that have any nonzero value: hitting the jackpot or matching 2-out-of-3.

EV = (1/35)(\$10) + (12/35)(\$3) = \$1.31.

Of course, you can never actually win \$1.31; any given ticket can only win \$0, \$3, or \$10. We see concretely that "expected value is not the value you expect."

Since the expected value is higher than the ticket price, this lottery game is very good for players, and very bad for whoever is using it to raise money. A player may lose in the short term, but long term you expect to come out about 30% ahead.

All of these calculations generalize to actual state lottery games, but counting up the number of ways to match, say, 4-out-of-6 on a 46-ball lottery is more complicated. We showed and briefly explained the combinatorial formulas for doing that, and then calculated the expected value for the Massachusetts Cash Winfall game (in the non-special weeks).

I pointed out that my usual winning was $9 on a $7 investment, slightly below the expected value. But that small difference is mitigated by the fact that I never lost at all.

Follow-Up Activity

We presented the following rules (the axioms for a projective plane), and asked teachers if they could draw a picture with a finite number of points that satisfied all of the rules:

1. Each pair of distinct points has a unique "line" between them.
2. Each pair of distinct "lines" intersects in a unique point.
3. Each "line" contains at least three points.
4. There exist at least three non-collinear points.

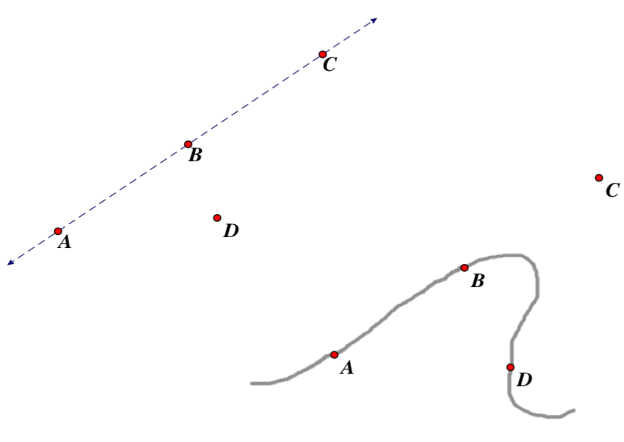


Figure 1. Lines

We emphasized that "lines" may not look like familiar straight lines from Euclidean geometry.

Teachers worked on this in groups for a while. The facilitators spent a lot of time with each group, clarifying the axioms and the task, and explaining that you can't satisfy all of the rules with just three points (or fewer): By Rule #3, they would all be on the same line, but Rule #4 says they have to be non-collinear. We were careful to draw curvy "lines" at every opportunity.

Later, participants presented arguments that we can't have just 4 points or 5 points. One group drew a picture with 7 points that they claimed satisfied all of the rules, and we had everyone check it carefully.

Tying It All Together

At this point, I introduced the Fano plane, which is the smallest possible example of a projective plane and has 7 points and 7 lines.

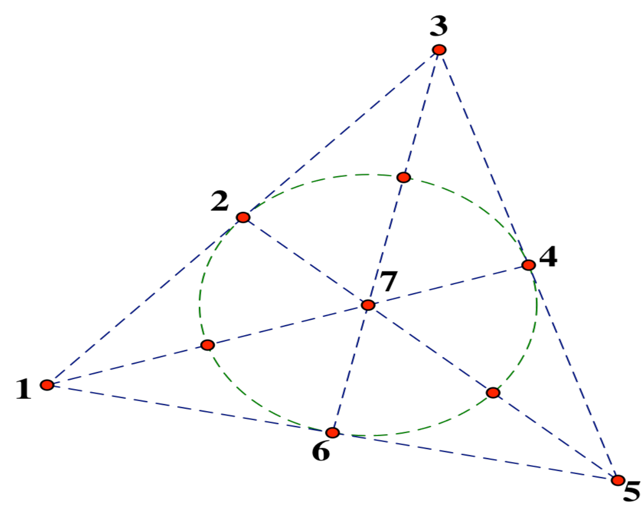


Figure 2. Fano plane

I pointed out that if we number the points from 1–7, then each "line" contains three points, as follows:

1-2-3, 1-4-7, 1-5-6, 2-5-7, 2-4-6, 3-4-5, and 4-6-7.

By Rule #1, every possible pair of numbers appears on a line somewhere, and by Rule #2 that pair appears on only one line.

So if we use these seven lines as lottery tickets, then no matter what the draw is, all three possible pairs are guaranteed to appear in the tickets. Either they are all on the same ticket (jackpot!), or they are on three different tickets (we match 2-out-of-3 three times).

We ended the session by regaling the teachers with an expanded version of the Massachusetts lottery story and answering any lingering questions.

Resources and handouts related to this article can be found at **www.mathteacherscircle.org/newsletter**.

Semiregular Tilings

A Topic Supporting Diverse Math Practices

by Nina White and Hanna Bennett

Questions about polygonal tilings of the plane can utilize a classical area of mathematics to highlight and connect middle and high school mathematics content standards, mathematical practices, and the nuanced nature of mathematical justification. Inspired by an MTC of Austin session on Escher-like tilings led by Altha Rodin, we ran a session on semiregular tilings of the plane at the Wayne County (MI) MTC last fall. Here we describe what mathematical features naturally arise at different points in the inquiry.

Semiregular Tilings

A *tiling of the plane* covers the (infinite) plane, without gaps or overlaps, using congruent copies of one or more shapes. We can consider special categories of tilings and ask what possibilities exist under those constraints. For example, we might ask that only one shape be used, that only quadrilaterals be used, or that only regular polygons be used. A *semiregular tiling* is a tiling of the plane with the following constraints:

- Two or more regular polygons are used.
- Polygons meet "edge-to-edge." That is, no vertex falls in the middle of an edge of another polygon—it always meets at other vertices. An immediate consequence of this is that all of our edges must be the same length.
- The pattern of polygons around every vertex is the same. (Note that the vagueness of the wording "the same" in this definition is intentional, as we discuss below.)

The overarching question of this session is:

Can you find all possible semiregular tilings of the plane?

Manipulatives are a useful tool for exploration as well as communication (see link to printable .pdfs at www.mathteacherscircle.org/newsletter); we suggest printing them on different colors of paper so that different shapes are easily distinguishable when used in a tiling together. The overarching question can be scaffolded by posing it as two consecutive questions:

Question 1: What possibilities exist around a single vertex?

Approaching this preliminary question creates a few intellectual needs:

1. the need to recall (or re-find) interior angle sum formulas. Using the fact that the interior angle sum of a triangle is 180º, there are several different justifications of the interior angle formula for convex n-gons that participants or facilitators might know. There is also a simple justification starting from the (equivalent) fact that the sum of the turn angles in any convex n-gon is 360º. Depending on the time available, this could be an opportunity to share some of these justifications.
2. the need to articulate what makes two vertex patterns the "same" or "different." This creates the opportunity to discuss rotational and translational symmetry, a middle school content area.
3. the need to systematically approach the question case by case. This third need is sometimes called *organization*, and is a centrally important mathematical problem-solving strategy (see Zucker article at www.mathteacherscircle.org/newsletter). If you try to align this particular skill with the eight Common Core State Standards for Mathematical Practice, you can find supportable overlap with at least six of the eight standards (nos. 1, 3, 5, 6, 7, and 8). If this practice of *organization* is made explicit in the course of the session, this alignment could make for an interesting discussion with teacher participants familiar with the CCSS Standards for Mathematical Practice.

To scaffold this question further, the facilitator can ask participants to first consider the case of triangles

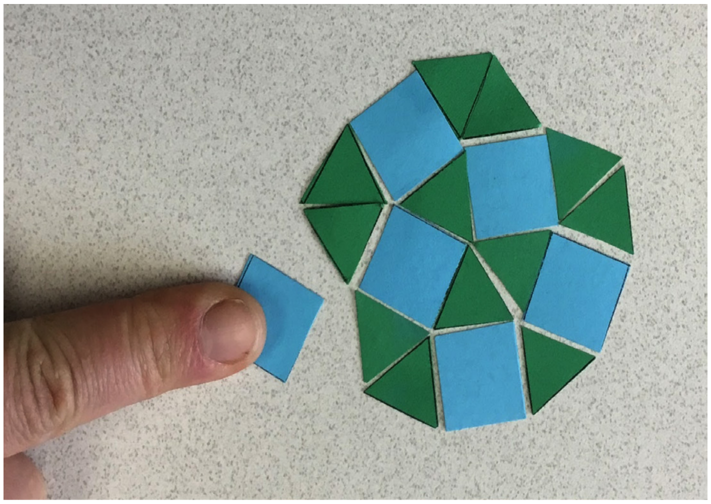


Figure 1. How can we place this next square? Can this vertex pattern possibly work on the whole plane?

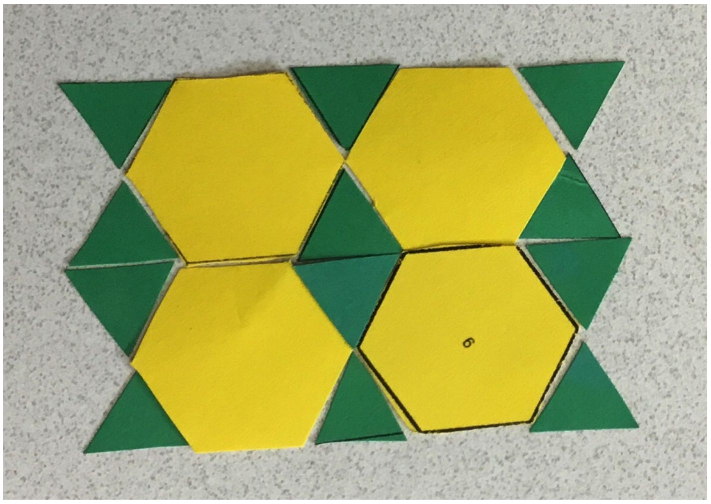


Figure 2. Is this a semiregular tiling? What is the vertex pattern?

and squares. That subquestion narrows the scope, allowing for more detailed involvement with needs (2) and (3).

Notation. For the purposes of record-keeping and communication, it may be useful to introduce a standardized vertex notation at some point, if teachers don't come up with their own. A vertex at which, cyclically, we see a "triangle, square, triangle, triangle, square" can be described as a 3.4.3.3.4 vertex.

Question 2: For each vertex pattern that works, can you continue the tiling forever?

Of all the possible tilings around a vertex, we focus on three that highlight different intricacies of mathematical justification.

Example 1: 3.3.4.3.4

This rich example will likely arise early in the exploration, because equilateral triangles and squares are familiar shapes, with familiar angle measures. It's well timed if it does arise early, because its complexity motivates participants to approach later vertex patterns with care. It naturally leads to two important mathematical questions about the general scenario at hand: 1) How do I continue the pattern beyond a single vertex? 2) How do I know the pattern will continue forever?

When participants work to continue this pattern, they will quickly find themselves having to make choices about whether to place a square or triangle along a given edge. It can be very productive (so don't try to stop it!) for some groups of participants to make

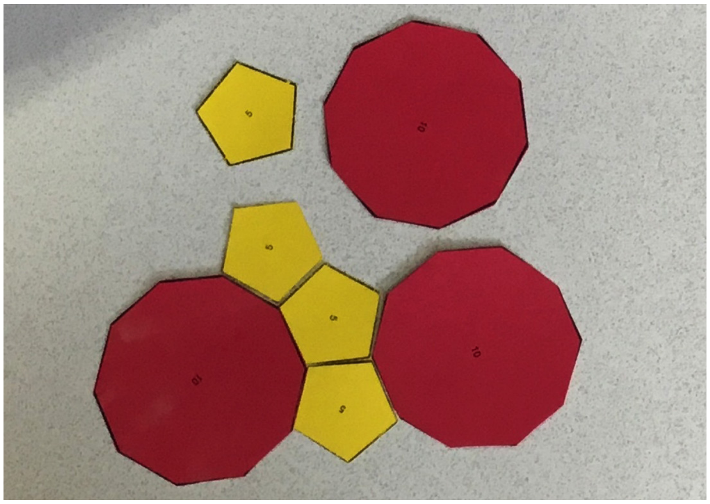


Figure 3. What goes wrong when we try to continue this vertex pattern? Is there a way to fix it? Why or why not?

What makes a good problem?

Watch MTC leaders Chris Bolognese, Fawn Nguyen, Paul Zeitz, and Joshua Zucker discuss this question in this Educator Innovator webinar: www.educatorinnovator.org/webinars/what-makes-a-good-problem

the wrong choices, leading to a tiling dead end that is not easily rectifiable without serious backtrack (see Figure 1). Even once participants have found the right choices to make in laying down the tiles, an example of a path that didn't work will hopefully motivate the need to truly convince ourselves that the pattern can continue forever.

Example 2: 3.3.6.6
This example is tricky because, starting with this vertex pattern, you can tile the plane. However, the most natural tiling that arises conflicts with the definition of semiregular tiling (see Figure 2). This is an opportunity to revisit the definition (and the importance of definitions in general), and to justify why the pattern cannot be continued in a fashion that fits our definition.

Example 3: 5.5.10
Although it is clear almost immediately that you won't be able to continue this vertex pattern (see Figure 3), articulating exactly why this is the case is a great exercise in mathematical argumentation.

Extensions

Changing the constraints in our definition allows for many additional questions. For example: What if we don't require our shapes to be regular? What if we don't require our vertex patterns to be uniform? What if we delete our "edge-to-edge" constraint?

This session could be part of a larger series of questions about tilings. See, for example, Rodin's article on Escher-like tilings in the Summer/Autumn 2014 *MTCircular*.

Shifting Gears
Approximations in Cycling
Michael Nakamaye

Nearly 17 years ago I moved to Albuquerque. The most important factor in this decision was the weather, propitious year-round for biking. I have been biking nearly my entire life, for fun, transportation, and exercise. In Albuquerque, what started as an innocent 15-mile round-trip commute has now evolved into a nearly 30-mile route, including an unnecessary-but-rewarding 1,000 vertical-foot climb.

When I learned that James Tanton had developed a Math Teachers' Circle session about biking, I watched the online video of the session with great enjoyment and fascination. Given a set of wobbly bicycle tracks that are the only clue left by an escaping thief, the session asks, "Which way did the thief go?" Tanton also presented a version of this session in Hawaii this past year, and so it seemed natural to build upon this in some way for my visit to Hawaii in April 2016. But where to begin?

After pondering the rogue thief's tracks for a bit, I thought to myself, "No decent bicyclist would ever leave tracks like that!" Indeed, part of what teachers discover in Tanton's activity is that the front wheel is traveling further than the rear wheel, which is very inefficient. In addition, the rider is losing her momentum by swerving. An experienced cyclist mainly uses the steering to make minor adjustments in the direction of travel; the main way a cyclist turns is by leaning in the direction of the turn.

After asking the teachers to think about these ideas briefly and recall some of the fun discoveries they made with Tanton, I moved the conversation in a different direction. How does a bike work? As you pedal, the bike moves forward. What causes this forward motion? The bike chain moves over each notch (or tooth) in the front ring near the pedals. As it does so, it also moves over one notch in the rear ring on the back wheel. So if there are x notches on the front ring and y notches on the rear ring, the bike advances by x/y revolutions of the rear wheel each time you make a full revolution of the pedals. The larger this fraction, the more difficult it is to pedal because you are advancing further per pedal stroke. For equivalent ratios, like 52:26 and 42:21, pedaling with 52 teeth (notches) in front and 26 in the rear will feel identical, and have the same impact, as pedaling with 42 teeth in front and 21 in the rear. A bike is like a *ratio machine*, giving the rider a concrete physical experience of different ratios.

It turns out that there are many other ratios to consider when riding a bike, all related to the gears:

- How many teeth are there usually on the different gears? Why?
- What is a comfortable cadence (number of pedal revolutions per unit time) for most riders?
- What is the range of speeds that you can comfortably go on a bike?

We then moved to a different type of question: Is there *duplication* between gears on a bike, i.e., two different gears whose ratios are equivalent? (The answer may depend on the bike, but is usually "no.") If not, which two gears are closest? (For the bike I brought for demonstration purposes, the two closest gears were 53:23 and 39:17.) Would you be able to feel the difference? (Yes, if you are working hard!)


The Math Teachers' Circle of Hawai'i (MaTCH) examines gears on a bicycle.

At this stage we "shifted gears," so to speak! I posed a rather abstract mathematical question: Suppose, as a cyclist, you wish to *feel* $\sqrt{2}$. What are some whole number gear values that form a fraction approximating $\sqrt{2}$?

Organizing our work into an ordered table with rows (r) of numerators (n) and denominators (d):

r	n	d
1	1	1
2	3	2
3	7	5
4	17	12
5	41	29

we see that by the fourth and fifth rows, the fractions are getting quite close to $\sqrt{2}$. To see why, notice that for any row $r > 1$, $n_r = n_{r-1} + 2d_{r-1}$, and $d_r = n_{r-1} + d_{r-1}$. Furthermore, for a given row r, if $(n_r)^2 - 2(d_r)^2 = 1$, then $(n_{r+1})^2 - 2(d_{r+1})^2 = -1$. Similarly, if $(n_r)^2 - 2(d_r)^2 = -1$, then $(n_{r+1})^2 - 2(d_{r+1})^2 = 1$. So, this table provides a quick and efficient way to produce lots of values of n and d satisfying $(n_r)^2 - 2(d_r)^2 = \pm 1$.

To see what this has to do with finding a ratio close to $\sqrt{2}$, note that we can rewrite this last equation as

$$\frac{n^2}{d^2} - 2 = \pm\frac{1}{d^2} .$$

The left side of this equation can be factored to give

$$\left(\frac{n}{d} - \sqrt{2}\right)\left(\frac{n}{d} + \sqrt{2}\right) = \pm\frac{1}{d^2} .$$

Since $\frac{n}{d} + \sqrt{2} > 1$ (it is close to 3), this means that

$$\left|\frac{n}{d} - \sqrt{2}\right| < \frac{1}{d^2} .$$

If $\sqrt{2}$ were a rational number, say $\sqrt{2} = p/q$, then the inequality above makes no sense once $d > q$, unless $n/d = p/q$, but this is not true for our choice of n/d. Somehow the fact that there are lots of rational numbers very close to $\sqrt{2}$ *tells us* that it is an irrational number.

Some numbers have extraordinarily good rational approximations, much better than $\sqrt{2}$. One example is π. Many people are familiar with 22/7, which is within just over one thousandth of π. Even more extraordinary is the fraction 355/113, which is within a few *ten millionths* of π. The subject of how close a number can be approximated with fractions of limited-size denominator has a very rich history and has occupied some of the best mathematical minds for centuries.

Other interesting observations and ideas that came up during the Circle session included:

- Participants can model the spinning of gears by forming two circles.
- Determining what speeds will be comfortable on a bike is a multi-layered ratio problem: You need to consider the ratio of front teeth to rear teeth, but also the ratio of the circumference of the tire to its diameter, and then your units of distance and time in order to find an answer.
- The technical details of how a bike works, though accessible, are not intuitive. (For example, it gets easier to pedal when you go to fewer teeth in the front but to more teeth in the rear!) Understanding how a bike works was, in a lot of ways, the biggest achievement of the session!

Participants model the spinning of a gear.

Picciotto Honored with Pólya Award

Picciotto

Henri Picciotto (**www.MathEducationPage.org**), a math education consultant and frequent guest leader at the AIM Math Teachers' Circle, received the 2016 George Pólya Award from the Mathematical Association of America (MAA). The Pólya Award is given for articles of expository excellence published in *The College Mathematics Journal*. Picciotto's article, titled "Square-Sum Pair Partitions," was co-written with Gordon Hamilton (**www.MathPickle.com**) and Kiran Kedlaya (University of California, San Diego). The article grew out of a blog post Picciotto wrote about the following problem: "Arrange the whole numbers from 1 to 18 into nine pairs, so that the sum of the numbers in each pair is a perfect square." In subsequent blog posts, Picciotto sought help as he tried to generalize the problem, and discussed his own experience with this problem as evidence that well-targeted hints can be a good thing. The blog posts inspired the resulting paper, which was published in the September 2015 *College Mathematics Journal*. The article can be found at **www.mathedpage.org/attc/in-addition/cmj-sq-sum-particitions.pdf**.

Borchelt, Faughn Present at ICME

Borchelt

Faughn

Nathan Borchelt and Axelle Faughn (Western Carolina University and Smoky Mountain MTC) were selected to present at the International Congress on Mathematical Education in Hamburg, Germany, in July 2016 (**www.icme13.org**). International Congresses are held every four years and offer a unique opportunity for mathematics educators from the around the world to discuss issues related to mathematics education. Participants interact with mathematics educators from around the world, listen to world-renowned scholars in mathematics and mathematics education, and take part in small, focused topic study groups on a wide range of topics. During the conference, Borchelt and Faughn co-led a workshop, co-presented a paper, and gave a poster presentation, all of which were focused on Math Teachers' Circles.

Lomas Publishes MTC-Related Lesson Plan

"Grid Paper Exploration," by Randy Lomas (Harvest Park Middle School and American Institute of Mathematics MTC), was recently published in the *California Mathematics Council ComMuniCator*. In the article, Lomas describes a middle school exploration of a favorite MTC problem that involves counting squares on a sheet of grid paper. Lomas began writing the lesson plan in summer 2015 at the AIM MTC summer immersion workshop, as part of a new program for veteran members of the AIM MTC. Lomas and five other classroom teachers paired up with mathematicians to create lesson plans with a focus on open-ended problem solving. The lesson plans were later classroom-tested by other members of the MTC. Lomas' lesson plan and Tatiana Shubin's MTC session on which it is based can be found on our website: **www.mathteacherscircle.org/resources/video-library/#grid**.

Lomas

Porath Receives Invitation to White House

Jane Porath (East Middle School and MTC Network Advisory Board) received an invitation from President Barack Obama to attend the Teacher of the Year event at the White House on May 3. Porath was nominated for this honor by the National Council of Teachers of Mathematics (NCTM). The White House issued a proclamation honoring "great educators" such as Porath for being on the front lines of progress: "As our nation has advanced on our journey toward ensuring rights and opportunities are extended fully and equally to all people, America's teachers…have helped steer our country's course. They witness the incredible potential of our youth, and they know firsthand the impact of a caring leader at the front of the classroom."

Porath acknowledged the honor in a press release from the Traverse City Area Public Schools: "I want to thank my colleagues and the school district for their support as I strive to be the best educator I can be for my students. I am proud and humbled to represent Traverse City Area Public Schools at the White House and to be recognized among the nation's top teachers."

Porath

Dispatches from the Circles

Local Updates from Across the Country

Colorado •

The **Northern Colorado MTC** ran its fourth residential week-long summer camp, July 10-15, at the YMCA in Winter Park, CO. This was the second joint summer workshop with **Rocky Mountain MTC**, with 35 middle school teachers participating.

Funding from the MAA-Dolciani Mathematics Enrichment Grant helped initiate a Student Math Circle. Members of the MTC helped tremendously to advertise the first Student Math Circle Summer Camp, May 31-June 2, with 52 local students attending.

Delia Haefeli (Winograd K-8 School) received the Teacher of the Year award from Greeley Chamber of Commerce and Success Foundation. She has been on the Northern Colorado MTC leadership team since its inception in 2011.

- Contributed by Gulden Karakok

Connecticut •

The **Fairfield County MTC** was awarded $500 from the Northeastern Section of the MAA for next year.

- Contributed by Hema Gopalakrishnan

Mississippi •

The **Mississippi Delta MTC** held four terrific meetings during its inaugural year. Circle member Penni Morgan, currently an elementary math coach, was admitted to a principal training program beginning summer 2016. Circle leader Liza Cope co-facilitated an MTC with Henri Picciotto on graph theory at the annual NCTM meeting in San Francisco. The circle used the remaining funds from its AIM seed grant to purchase a 3D printer. Circle member Mary Kline brought her 6th grade class to Delta State University to make an optical illusion spinner after researching the math connections to 3D printing and optical illusions in their Scholastic magazine. The Circle held another successful Domino's Pizza Dough Raiser to raise funds for future meetings.

- Contributed by Liza Cope

New Mexico •

James Taylor (**Santa Fe MTC**) presented a luncheon talk at the New Mexico Mathematical Association of Two-Year Colleges Conference, May 20-21, in Los Lunas, with 90 participants from New Mexico colleges, El Paso, and Flagstaff. He introduced math circles with a mix of history, description, and an activity on Brussels sprouts, graphs, topology, and the Euler characteristic. He also ran a two-hour circle session of Liar's Bingo.

- Contributed by James Taylor

New York •

Japheth Wood (**Bard Math Circle**) won the distinguished service award from the MetroNY section of the MAA. The award cited his work with math circles at the local, regional, and national levels (**http://sections.maa.org/metrony/pastwinners)**.

- Contributed by Japheth Wood

Oregon •

The **Portland MTC** is excited to have been awarded a seed grant from AIM. They are holding a kick-off workshop on August 16 -17, eager to explore math together!

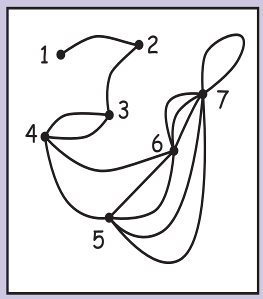
- Contributed by Kara Colley

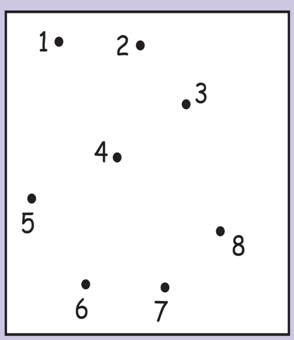
Pennsylvania •

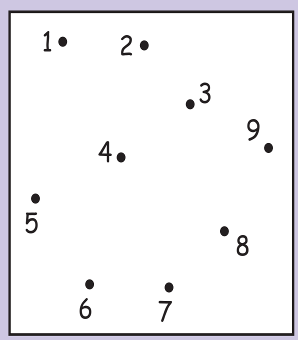
The William Penn Foundation recently awarded the University of Pennsylvania Graduate School of Education (PennGSE) a grant to support mathematics instruction in Learning Network 2 of the School District of Philadelphia. The **Philadelphia Area MTC** has been named as a consultant on this grant, to further its work with teachers on mathematical problem-solving and interpreting and implementing the Standards for Mathematical Practice of the Common Core. Additional funding for PAMTC's effort has been pledged by AIM and a private, Philadelphia-based donor.

- Contributed by Joshua Taton

MATH WITHOUT WORDS







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