Julia Robinson Mathematics Festival

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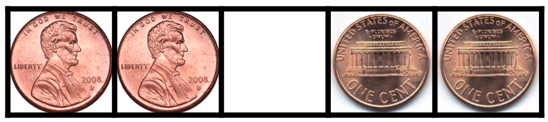
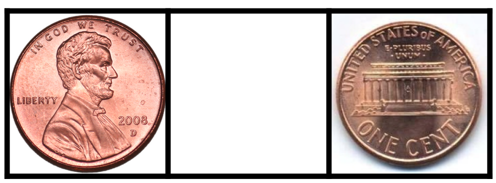
Frogs and Toads - Solution

Solving an easier problem is a huge key to making progress with Frogs and Toads. With the 5 by 5 board, there are a lot of moves in the solution, and lots of places where one false move leaves you stuck many moves later without any easy way to see just where you went wrong. However, if you simplify things to 1 by 5



or even 1 by 3



then you can solve those problems easily. Solving an easier problem is always a good strategy for getting started on anything, but as we'll see, in this problem it is even more powerful than usual.

In solving the 1 by 3, which takes only 3 moves, we can develop a few possible notations for these moves. In the linear problems, at each step there's at most one frog and at most one toad that can move, so we can encode the process as FTF, telling us which kind of animal moves at each step. (Of course TFT works just as well.)

Another possible encoding is by what kind of move we use. We could encode this as SJS, with S for slide and J for jump. This can be ambiguous, though: what if two animals could slide to the same spot? Here that's advantageous, as this one solution captures both FTF and TFT, but are there places where this ambiguity is problematic?

We could also encode by the position from which the animal moves, so our solution could be represented as 132, or if we want 0 in the middle of our line, as –1 1 0. What about encoding by the position that the animal moves to? There are a lot of patterns to uncover among these various encodings as the board gets bigger.

Once we've mastered the 1 by 3 and then the 1 by 5, we can use those solutions to build up a solution to the 5 by 5. This is a great example of how a solution to an easier problem can directly build the full solution to a much harder problem! If you solve the 5 by 1 vertical puzzle in the middle column, at various moments the empty space will be at each spot in that column, and you can solve a 1 by 5 horizontal puzzle. Hence, since the 1 by 5 takes 8 moves, the whole puzzle will be solved in 48 moves. Can anyone prove that it can't be solved in fewer moves?

By similar logic, the $2n+1$ by $2n+1$ board can be solved in $(2n+2)((n+1)^2 - 1)$ moves.