Puzzles, Bands, and Knots

Mathematics Teachers' Circle of Austin

The University of Texas at Austin

February 26, 2015

Jennifer K. Mann Austin, Ph. D.

Each participant needs:

- 4 rectangular strips
- 4 long pieces of tape (~3” each)
- 1 Blue knot/unknot paper
- 1 pair scissors
- 1 Pythagorean Puzzle
- 1 rope

Pythagoras and His Hypotenuse

How a puzzle leads to a proof of one of the gems of mathematics

The knowledge at which geometry aims is the knowledge of the eternal.

PLATO

Pythagorean Theorem

If a triangle is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

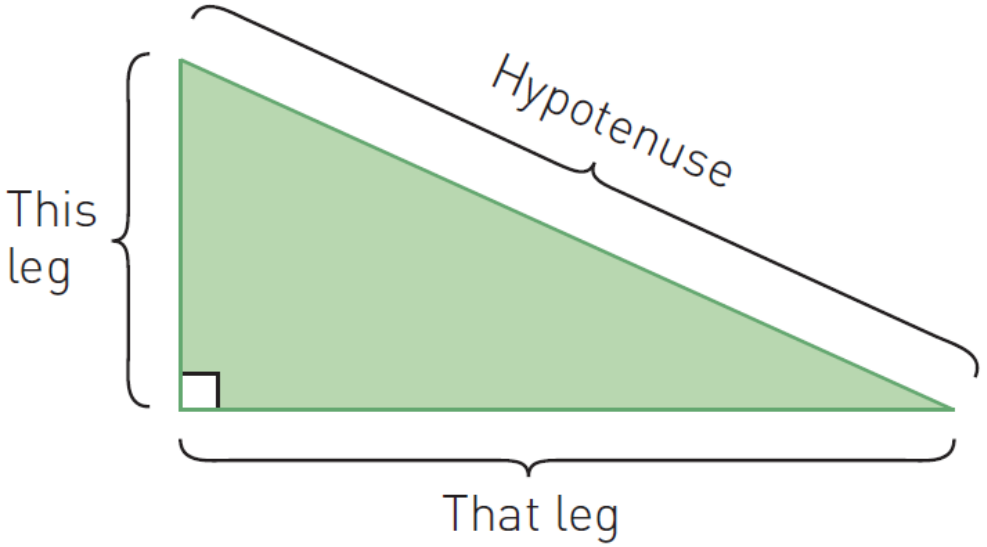


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Pythagorean Theorem

$$a^2 + b^2 = c^2.$$

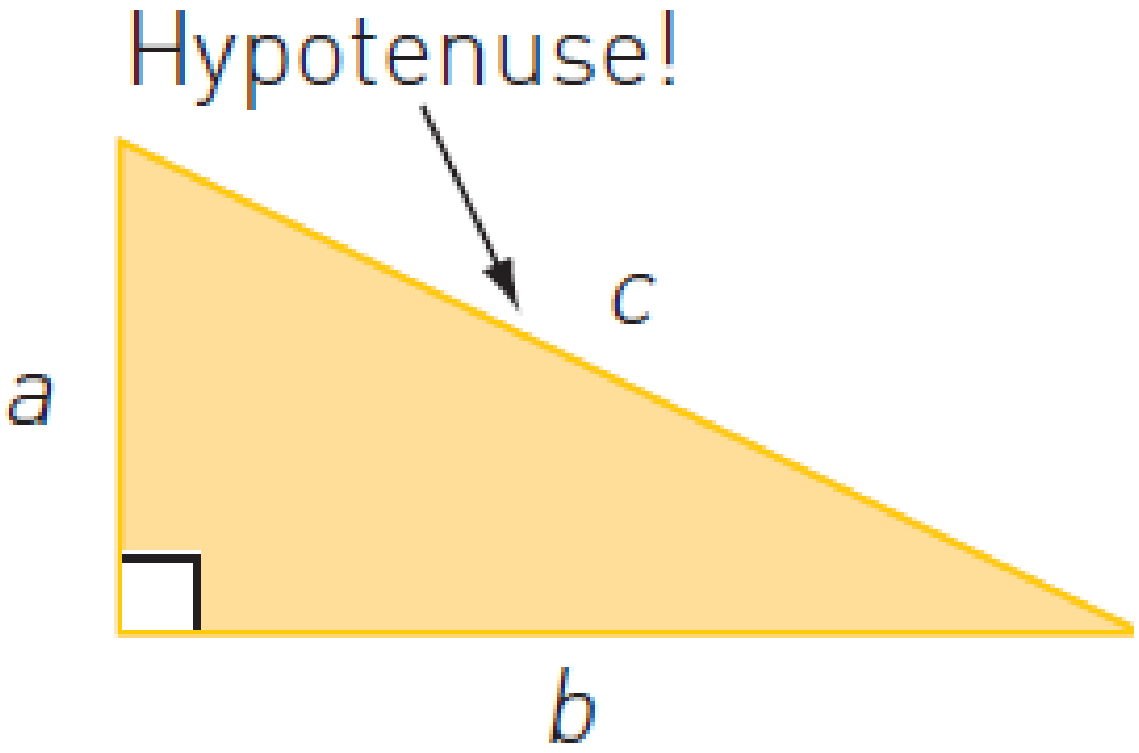


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

The Puzzle Proof

A proof of the Pythagorean Theorem

The Puzzle Proof: Step 1

What is the total area of the 5 pieces?

Can you construct a single large square with side length equal to c?

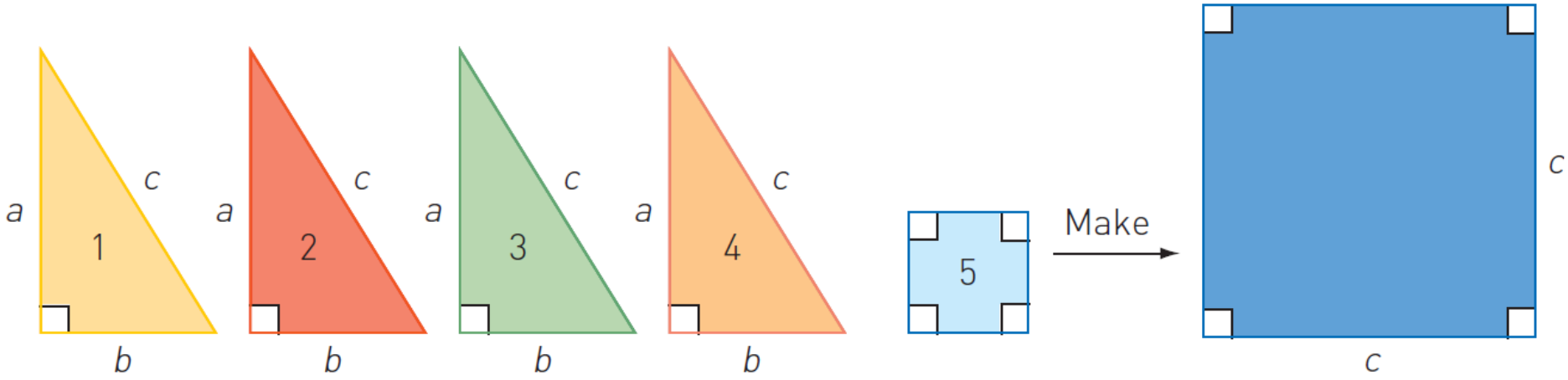


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

The total area of the 5 pieces is c^2.

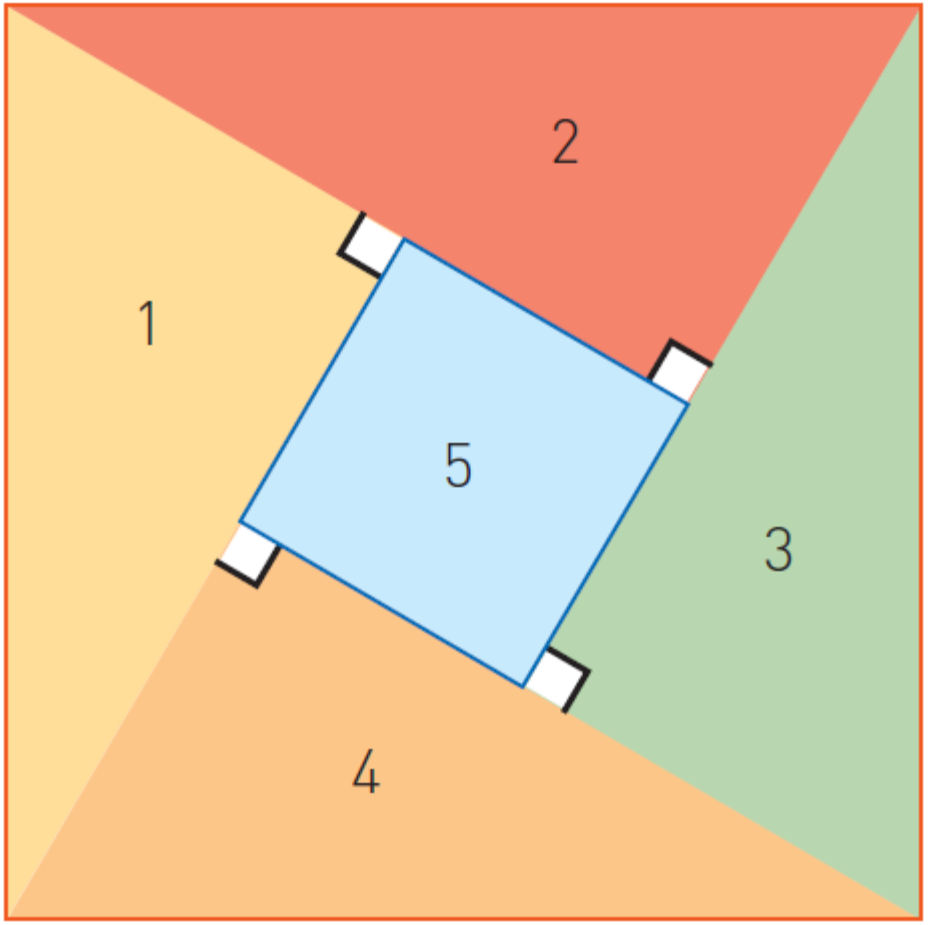


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

The Puzzle Proof: Step 2

Make two squares.

Keep an open mind to all possibilities.

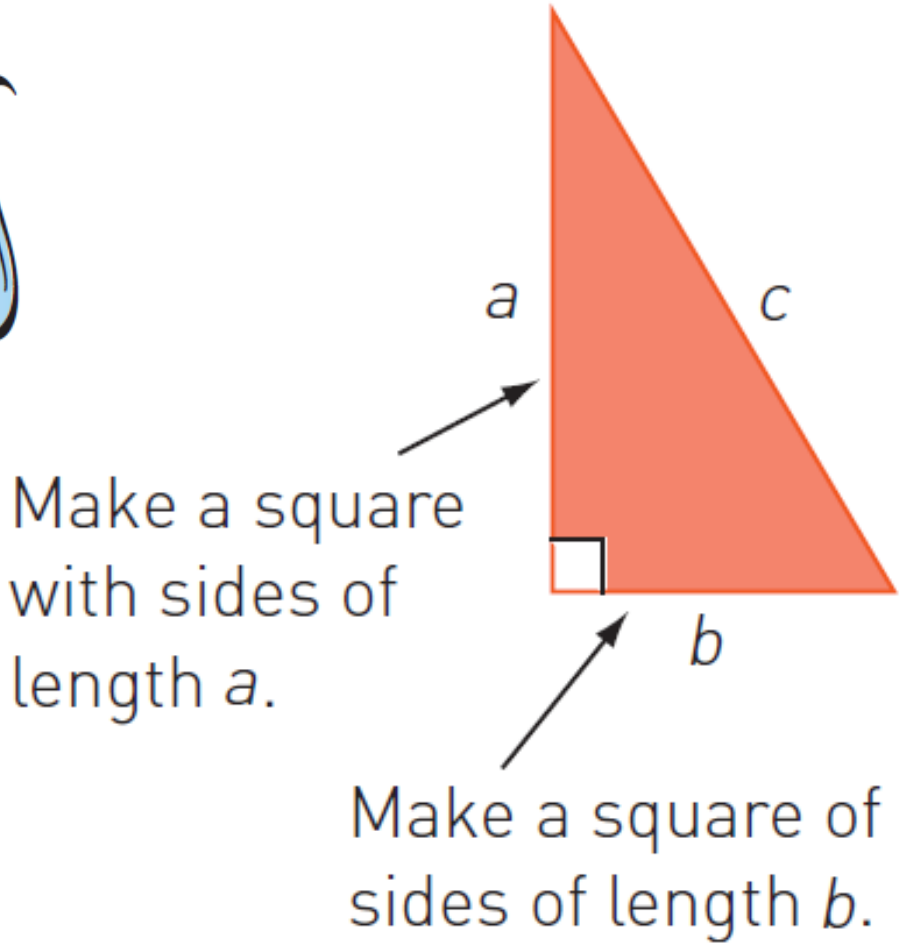


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

The total area of the 5 pieces is a^2+b^2.

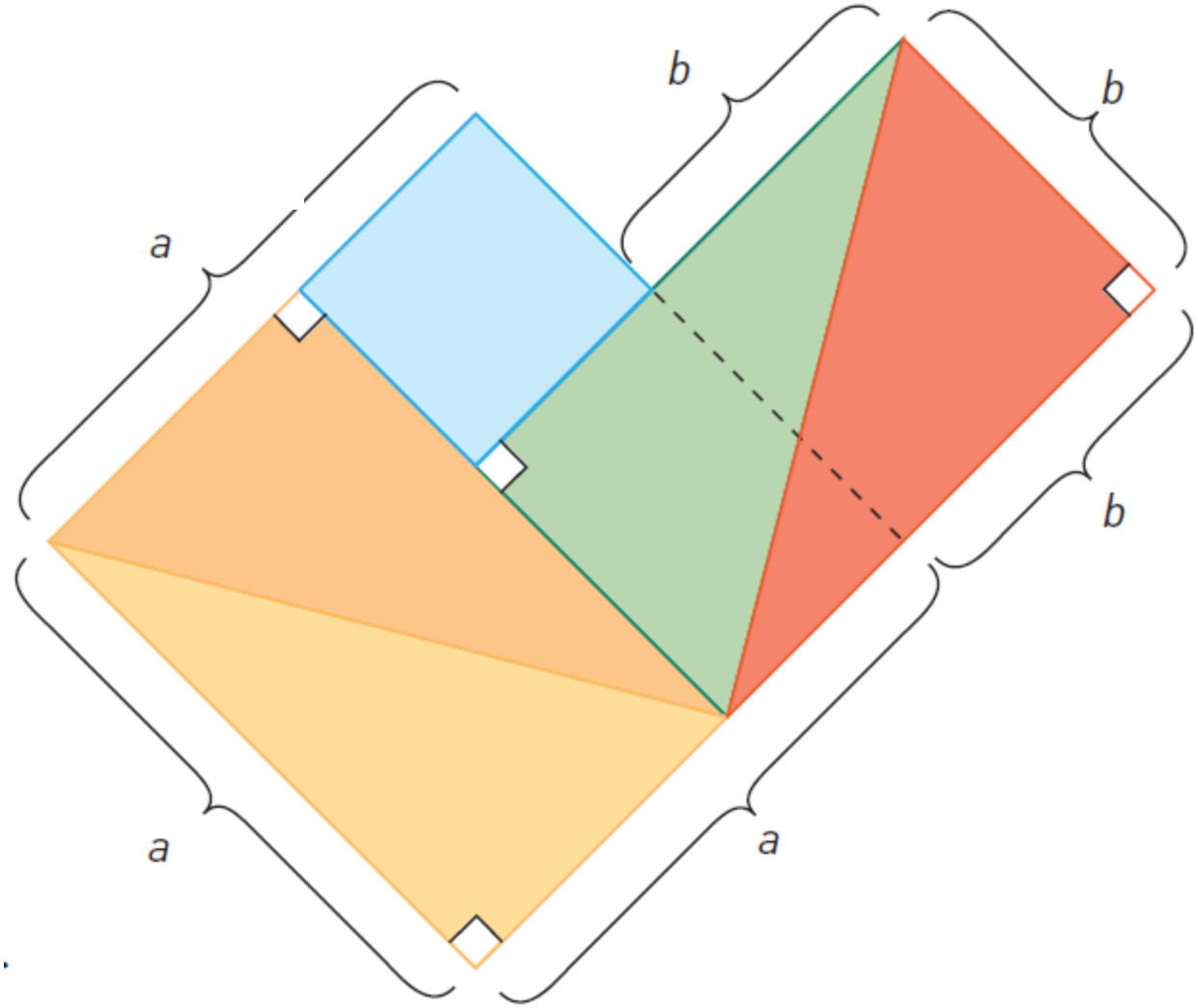


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Thus, we have $a^2 + b^2 = c^2$.

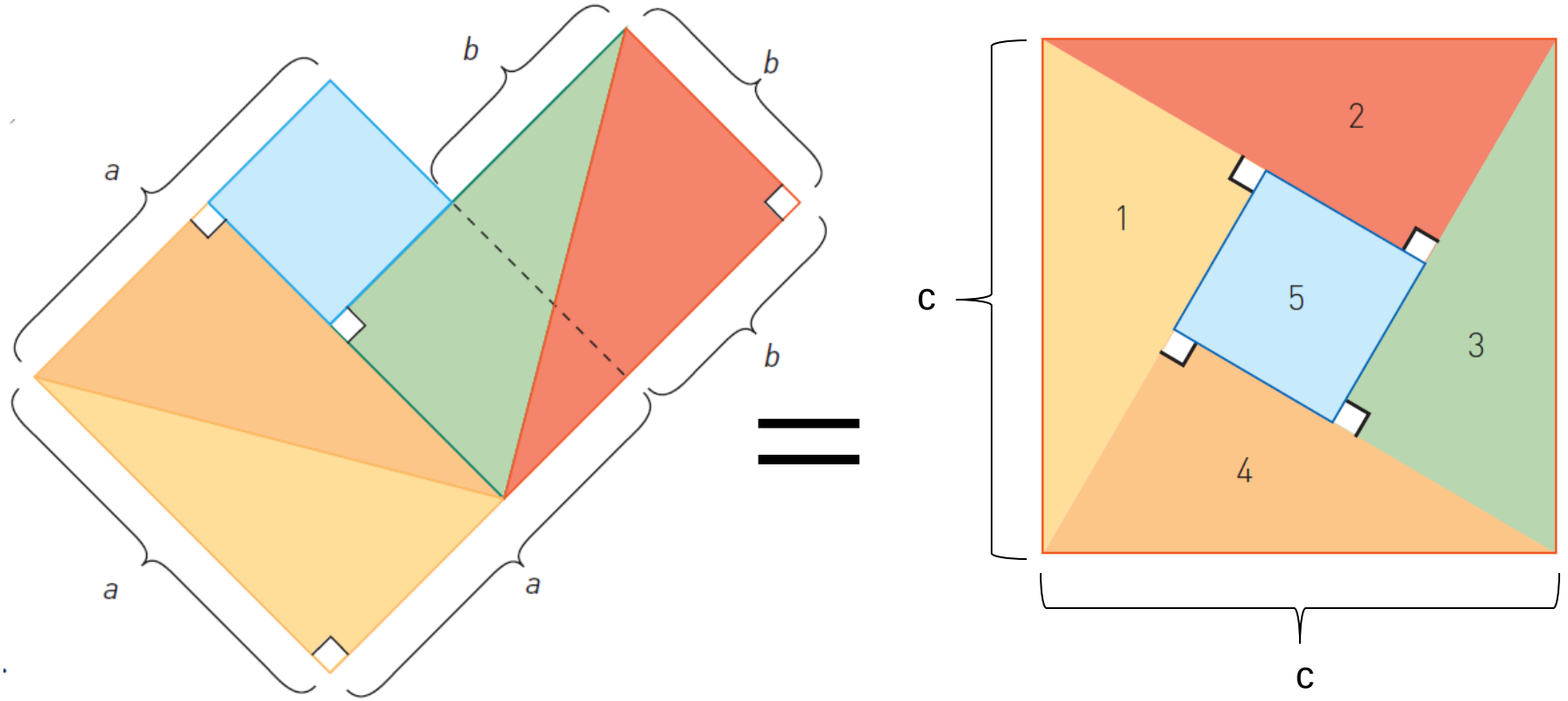


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

The Band That Wouldn't Stop *Playing*

Experimenting with the Mobius Band

Make guesses!

Question of the Day

Take a strip of paper and tape the short ends together to make a loop. How many pieces do you get if you cut the loop down the middle?

Do NOT actually do this experiment!
Just think about it.

The Mobius Band

How do you make a Mobius band?

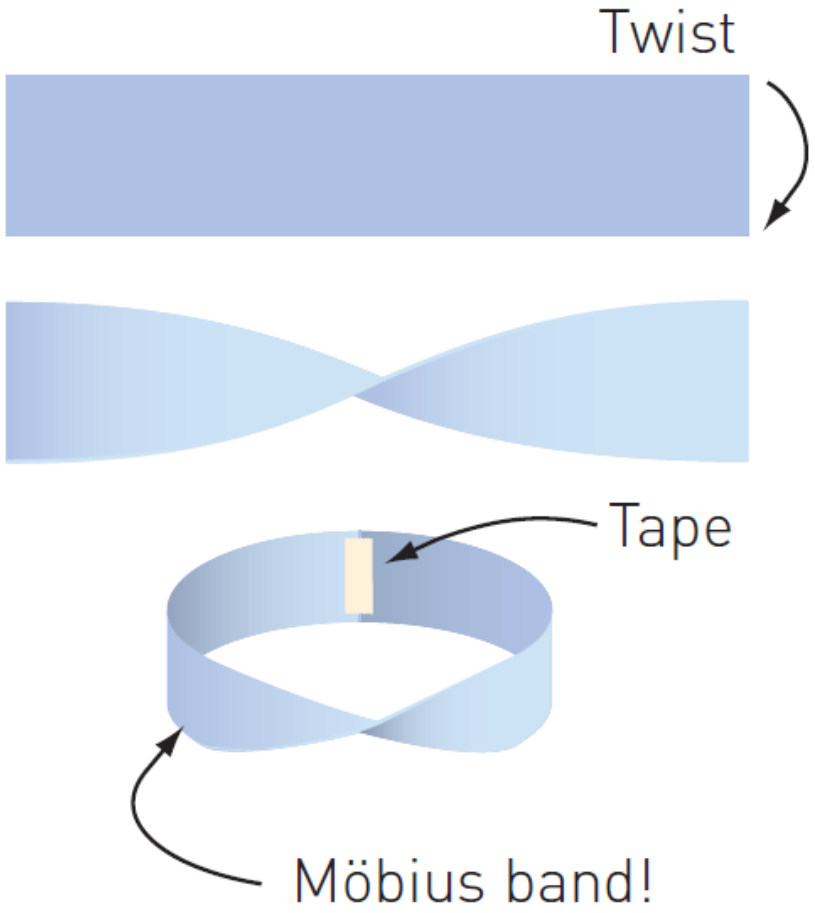


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

The Mobius Band

How many sides does a Mobius band have?

Trace along the center of the band with a pencil. What do you notice?

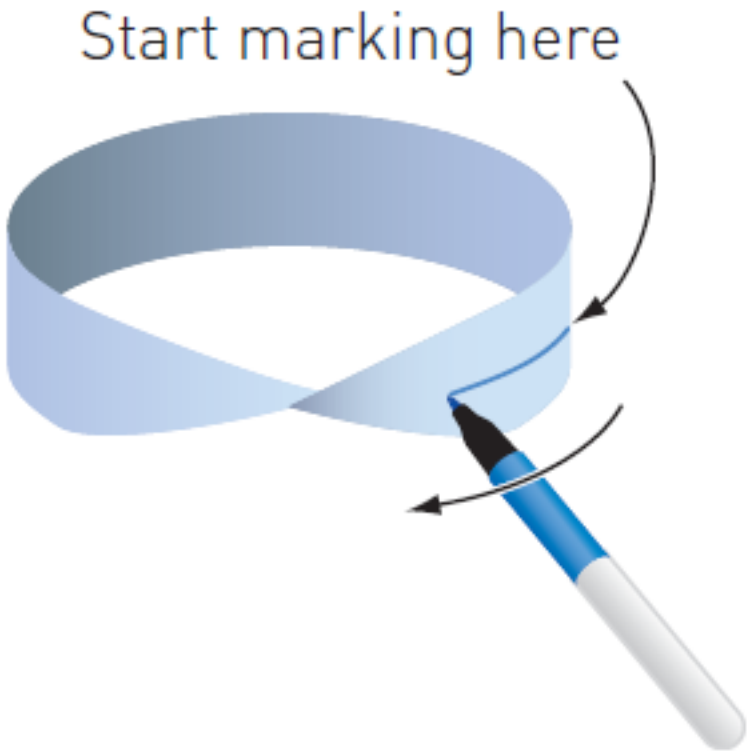


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

A Mobius Band is One-Sided!

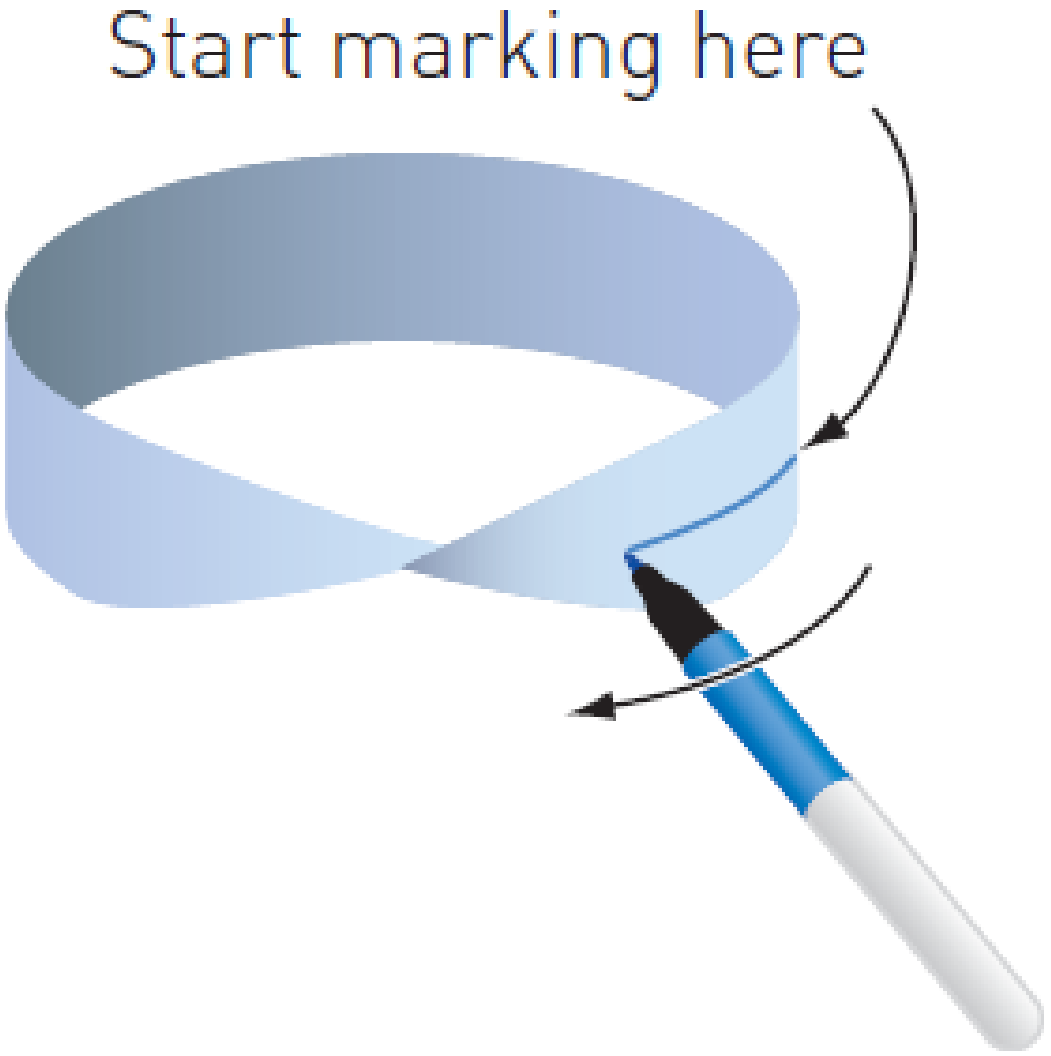
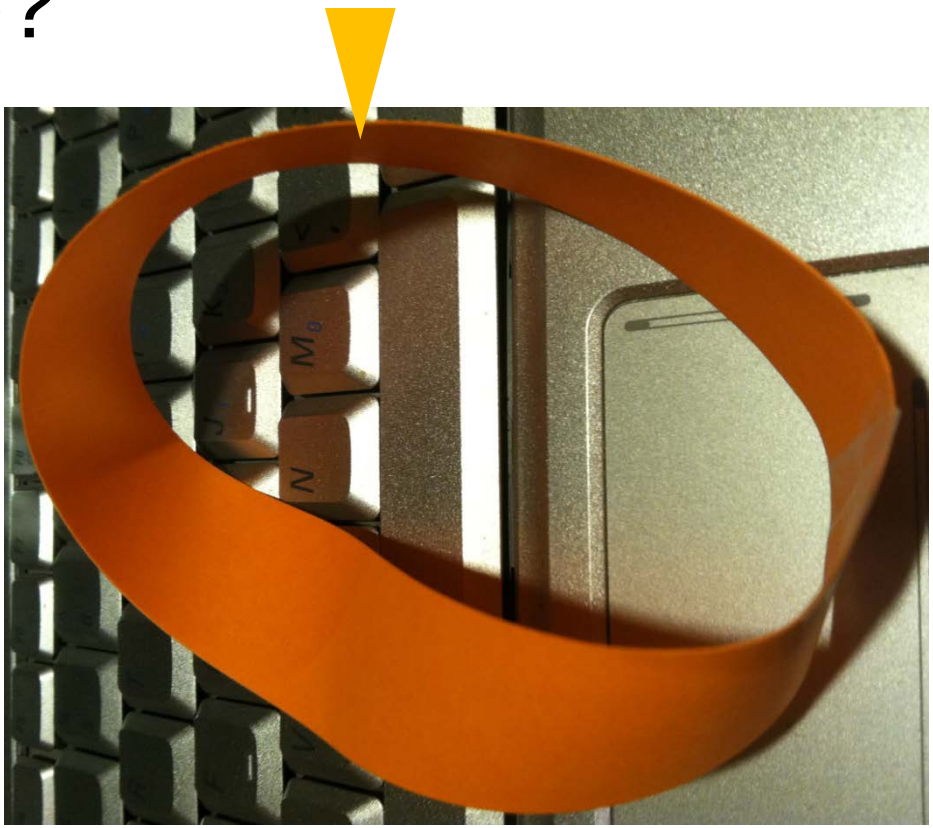
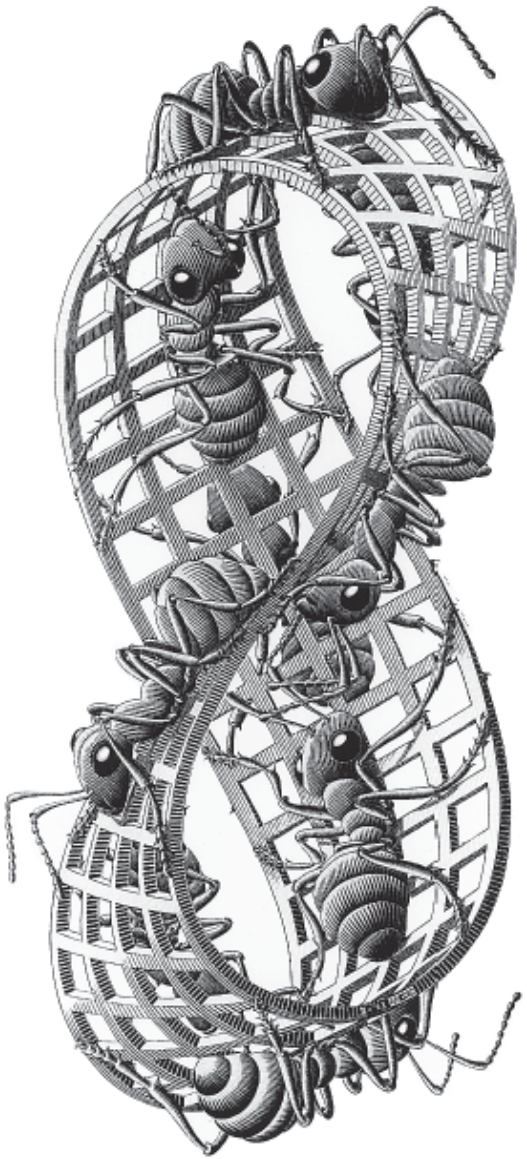


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

How many edges does a Mobius band have?

Make a notch on one edge. Trace your finger along the edge. What do you notice?





A Mobius Band has only one edge!

Mobius Strip II, by M.C. Escher. (© M.C. Escher's Mobius Strip II © 2004 The M.C. Escher Company—Baarn—Holland. All rights reserved.)

Twists versus Sides (or Edges)

How does the number of twists in a band relate to the number of sides and to the number of edges?

Half-Twist ↔ One Side

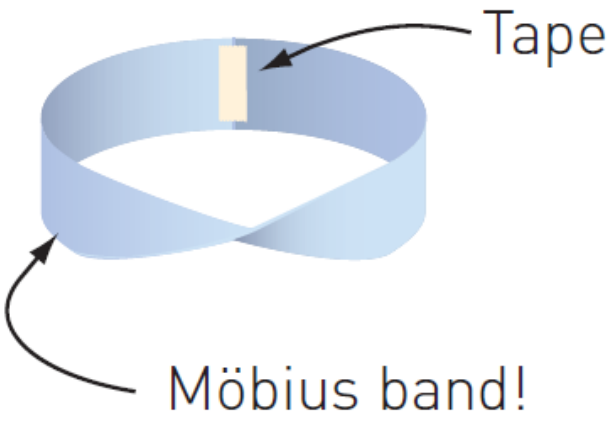


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Other Mobius Band Explorations!

Exploration A: Cut lengthwise down the center core of the band. What do you see?

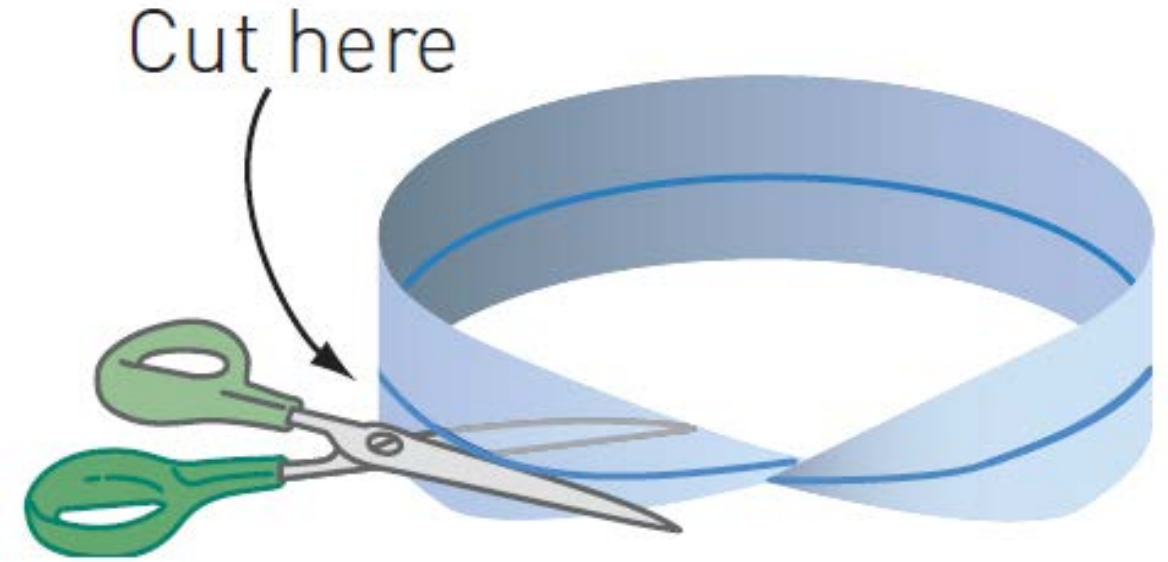
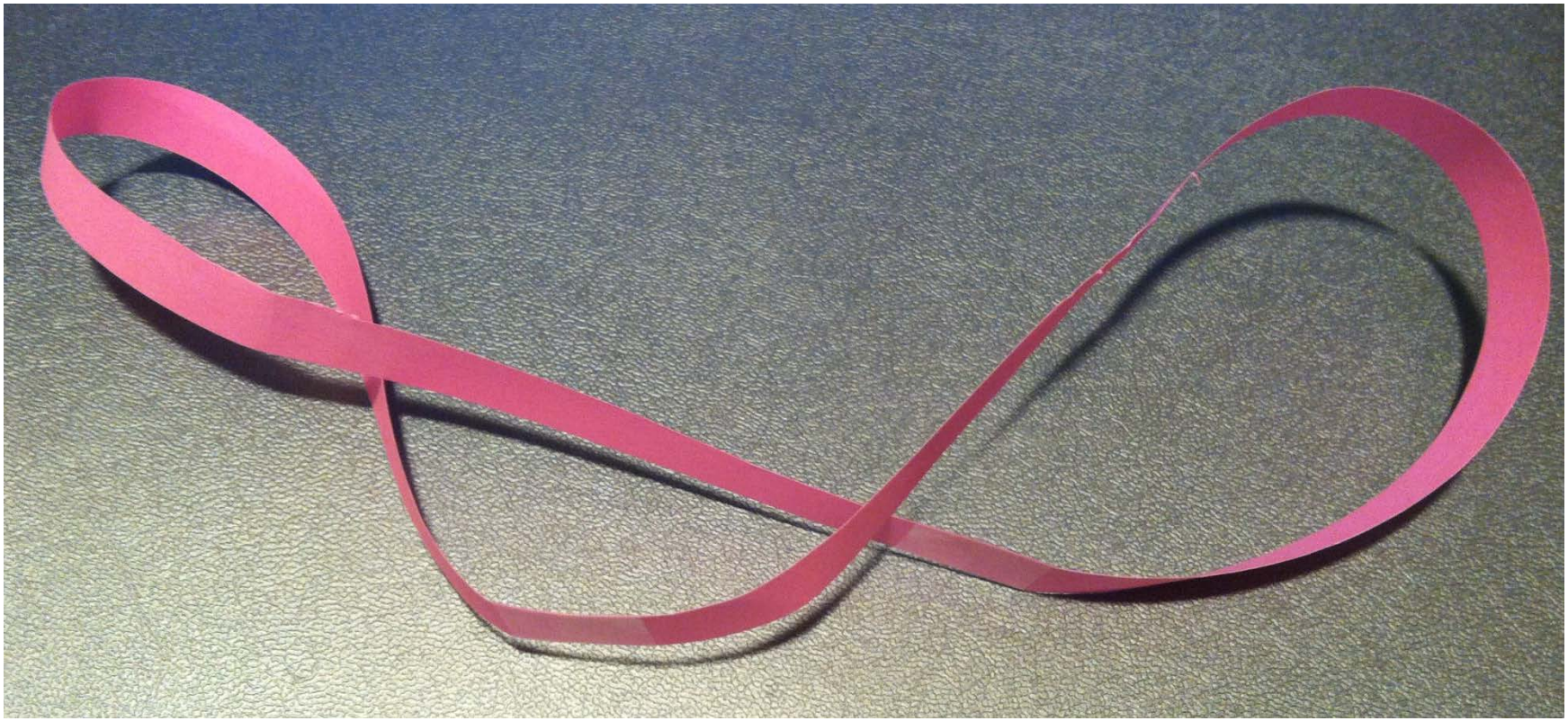


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Exploration A Result

1 band that is twice as long as the original band and now has 2 half-twists



Mobius Band Exploration B

Make another mobius band and cut by staying close (about 1/3 of the way) to the right edge. What do you see?

Cut, staying $\frac{1}{3}$ inch away from the right edge of the strip with the scissors.

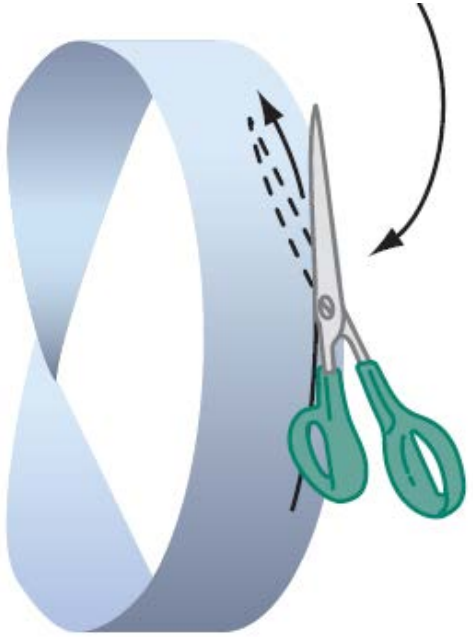


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Exploration B Result

The result should be 2 bands! One band that is twice as long as the original band and now has 2 half-twists. The other band will be the same length as the original and have 1 half-twist (so this band is also a Mobius band).

Exploration B Result

Mobius Band Exploration C

Make a band with three half-twists. Cut lengthwise down the center core of the band. What do you see?

Exploration C Result

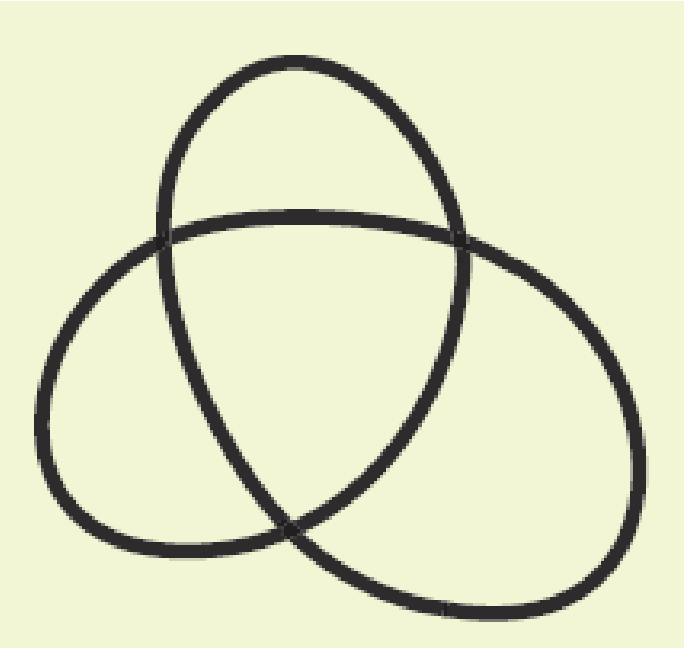
Knots and Links

Untangling Ropes and Rings

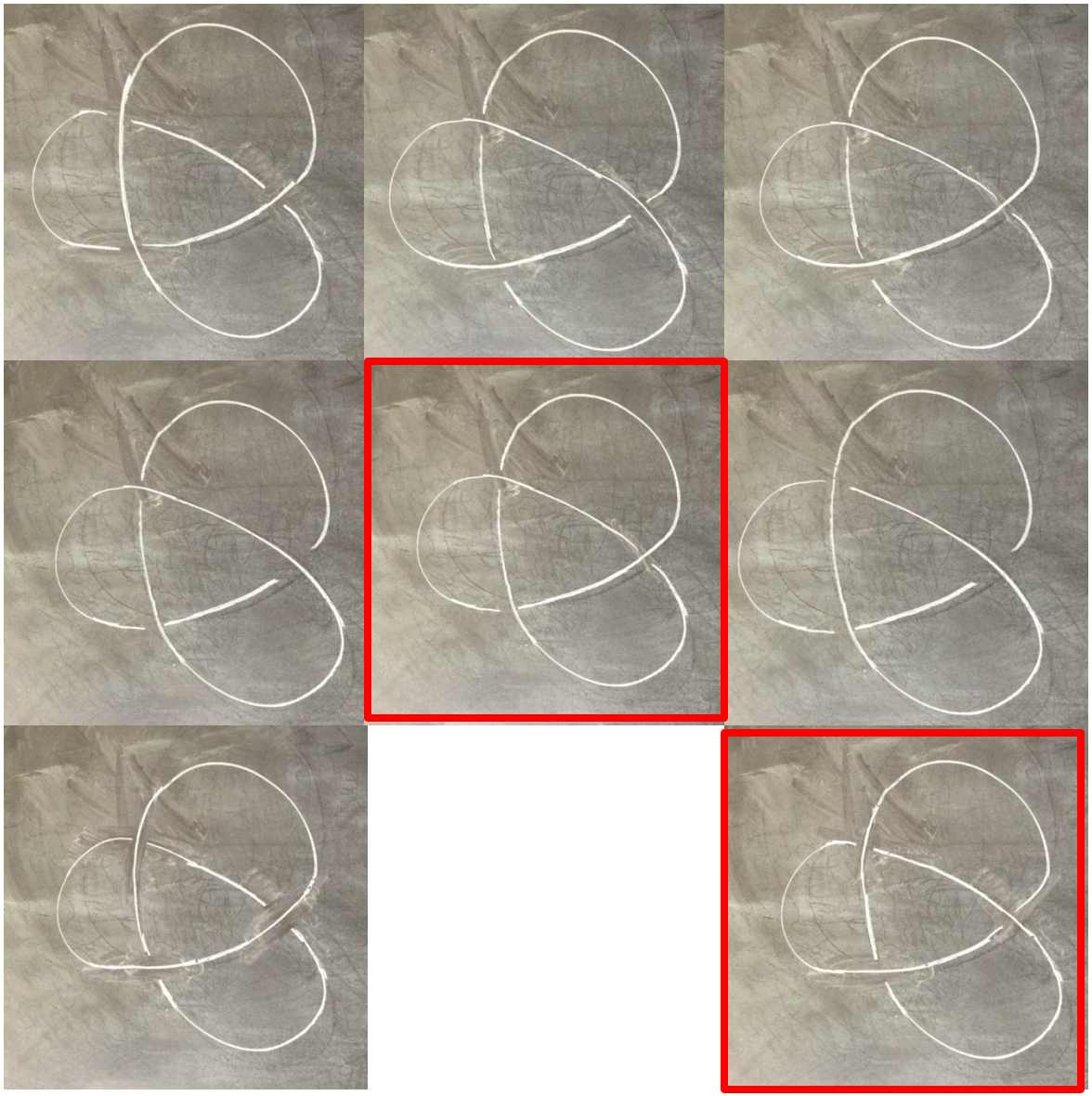
Experiment to discover new insights.

Knotted Rope?

Suppose a rope is lying on the floor in the pattern shown below and the rope is too far away for you to see how it crosses itself. What is the probability that the rope is knotted?



Knotted Rope?



What is a knot?

Determine which of the following are mathematical knots.

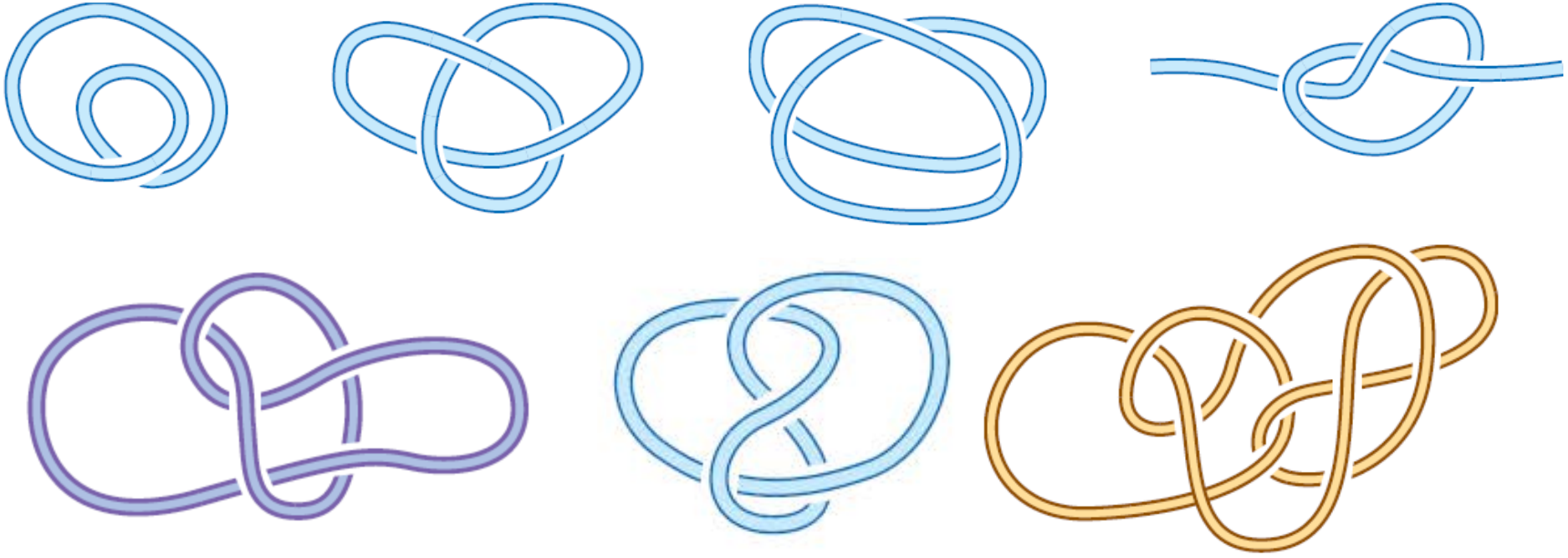
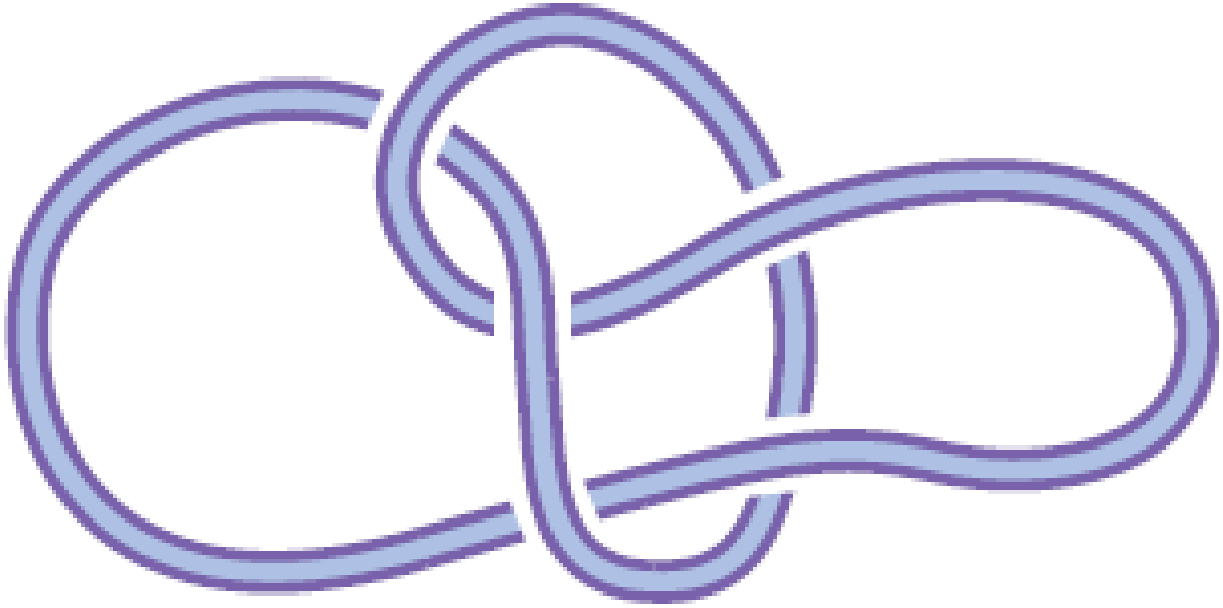


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

When is a knot not a knot?

Draw intermediate steps to show how to untangle the pictured knot to become the unknot.



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When is a knot not a knot?

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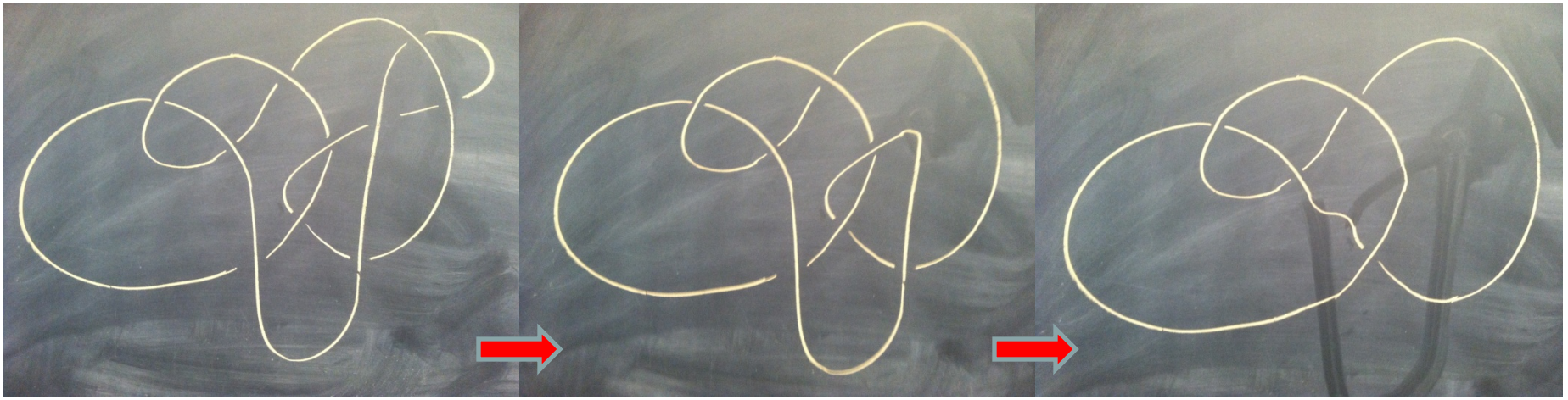
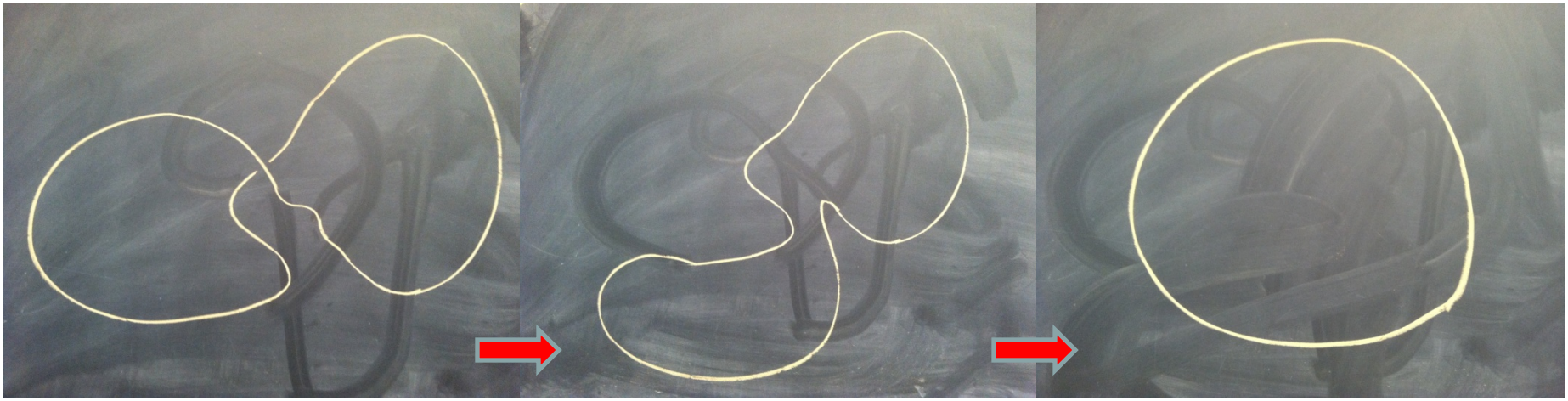
When is a knot not a knot?



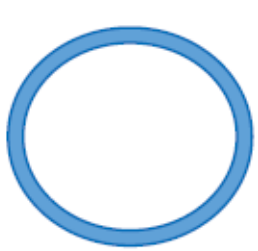


Human Knot or Unknot?

Stand in a circle of 7-8 people. On the count of three, everyone puts their hands in the circle and grabs someone else's hands at random. Without unhanding anyone, have the group move around and attempt to unknot. Were you a knot or not?

Crossing Number?

Determine the number of crossings shown in the diagrams of each of these mathematical knots.



Unknot

Trefoil knot

Figure-eight knot

Square knot

Granny knot

Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Crossing Numbers

Suppose you are given pictures of two knots. If they have a different number of crossings, then must the knots be different knots?

If so, explain; if not, provide different pictures of the same knot with different numbers of crossings.

Unknotting Number?

How many crossing changes are needed to unknot each of the knots below?

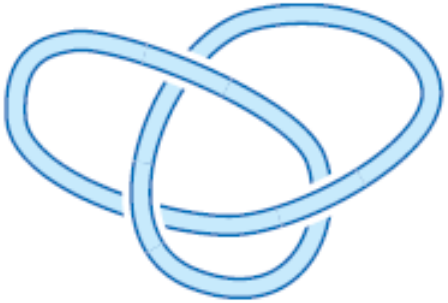


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Human Knot Experiment

Find five fellow students. The five of you join hands as shown.

Without unhanding anyone, have the group move around and attempt to unknot.

Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Human Knot Experiment

Once your group has tried in earnest to unknot, have everyone regroup and take their original positions.

Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Human Knot Experiment

Now, switch two crossings, as shown, and try to unknot.

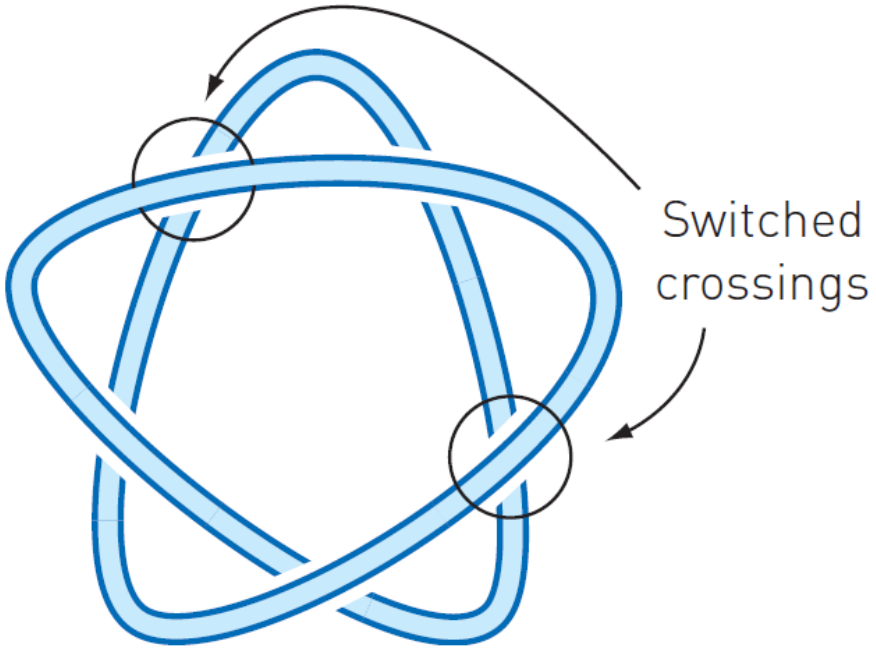


Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

Human Trefoil Experiment

What is the minimum number of people you need to make a human trefoil knot?

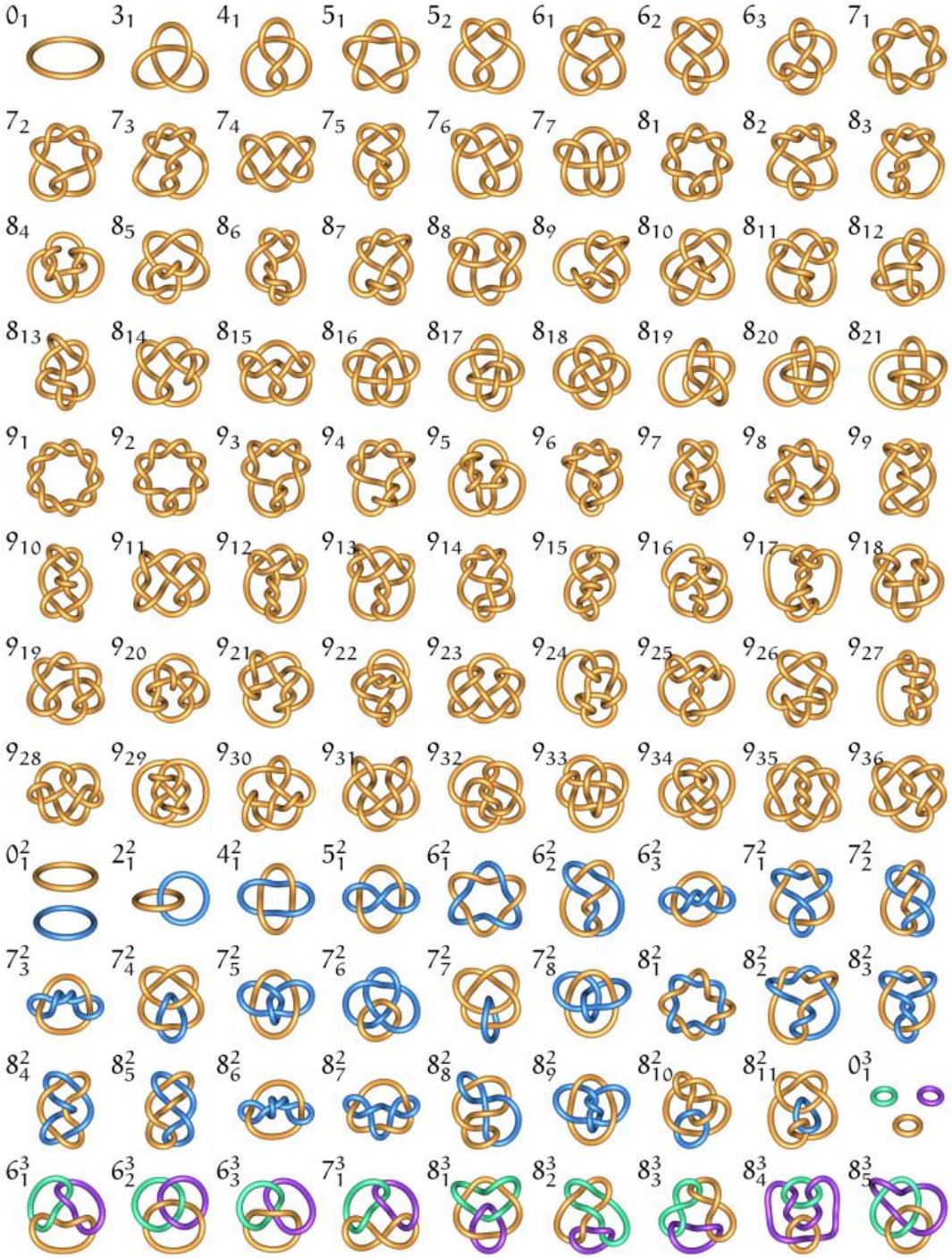
Trefoil knot

Image adapted from The Heart of Mathematics: An Invitation to Effective Thinking, 4th Edition by Edward B. Burger and Michael Starbird

A Knot Zoo

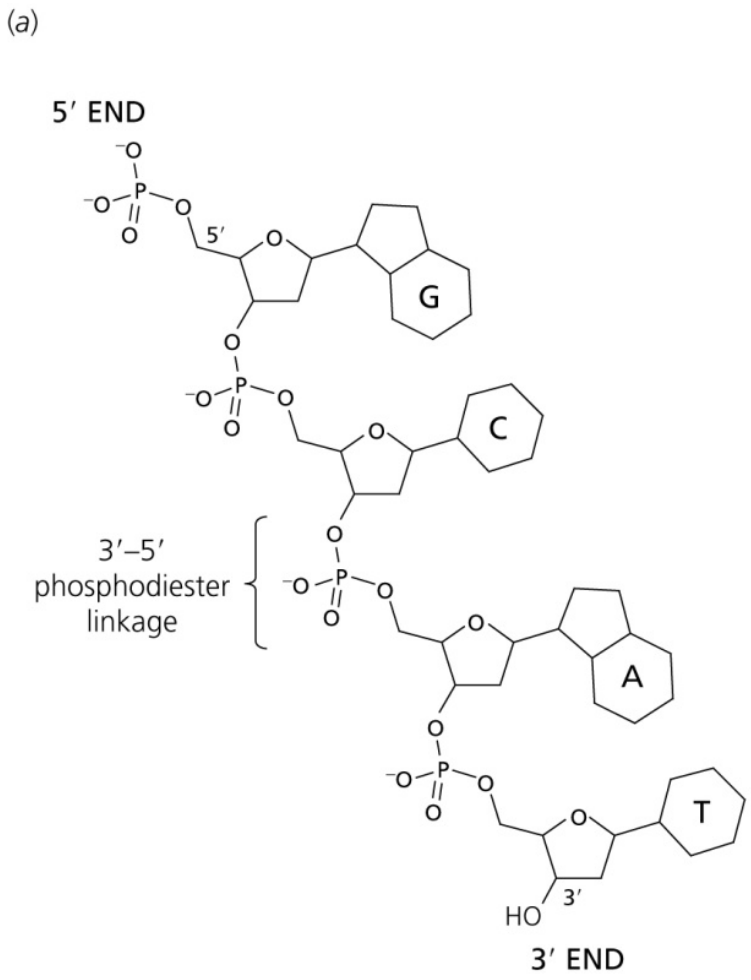
By Robert G. Scharein

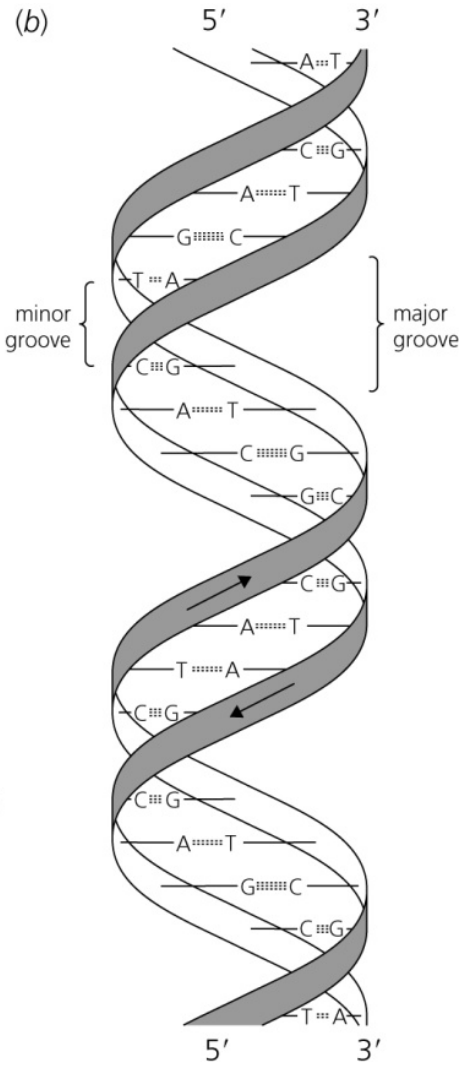
http://www.pims.math.ca/knotplot/zoo/



What is a DNA?

What is a DNA Knot?





DNA Structure

(a) polynucleotide chain

(b) B-DNA ribbon representation

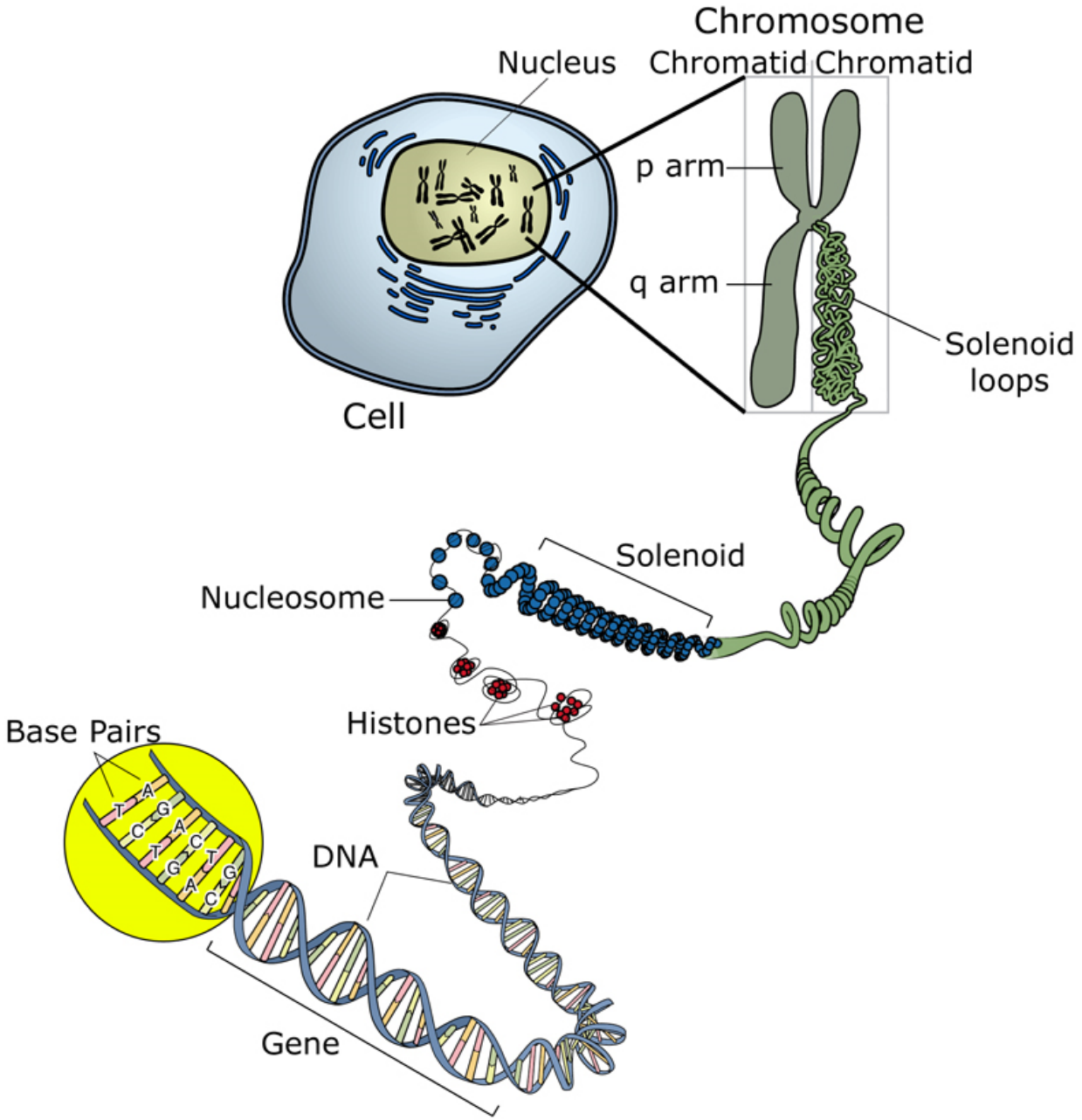
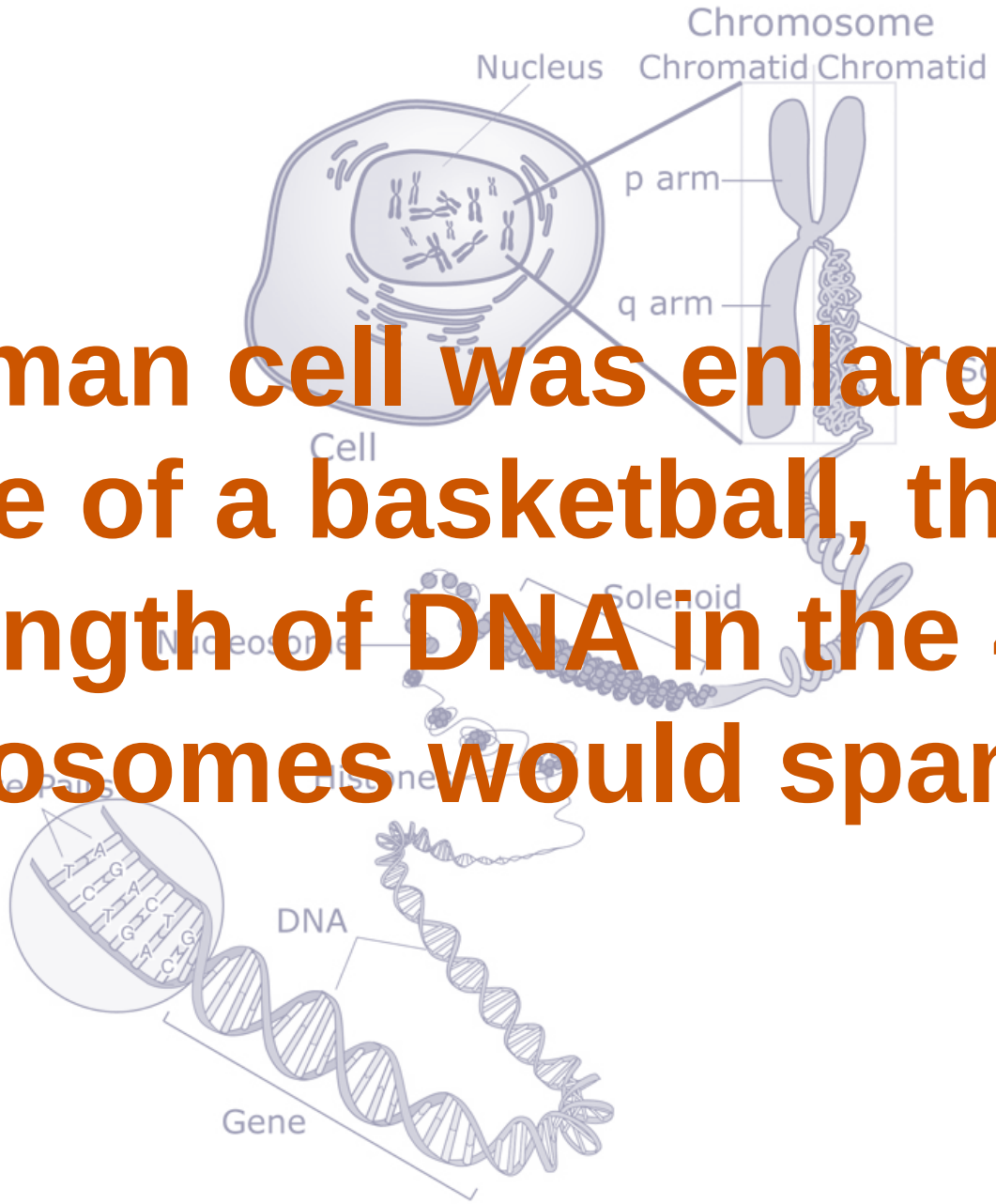


Image adapted from: National Human Genome Research Institute.



If a human cell was enlarged to the size of a basketball, then the total length of DNA in the 46 chromosomes would span 150 miles!

Image adapted from: National Human Genome Research Institute.

“Topology affects virtually every nucleic acid process that requires the double helix to be opened or moved within the cell.”

Deweese JE, Osheroff MA, Osheroff N., Biochem Mol Biol Educ. 2008;37(1):2-10.

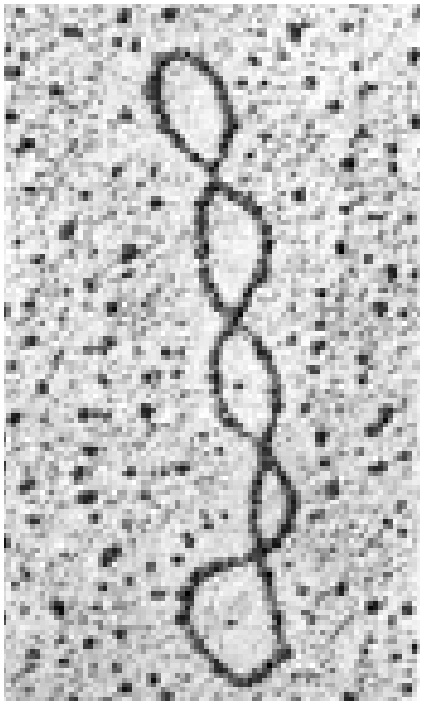
Will DNA ever form a Mobius band?

What was the result when you cut down the center of the Mobius band?

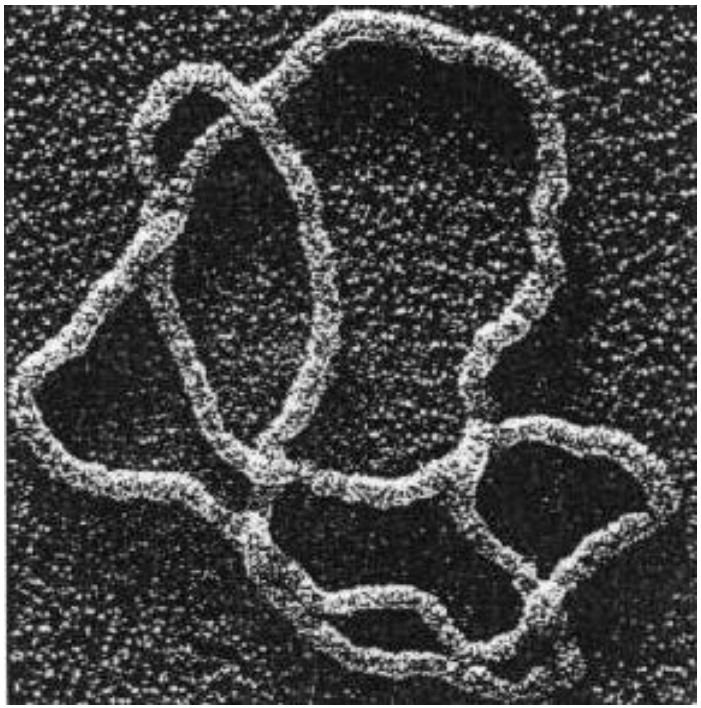
Now take a paper ribbon & add 2 half-twists before you tape the ends. What is the result when you cut down the center of the band?

If DNA does not form Mobius bands, then what different topologies can DNA have?

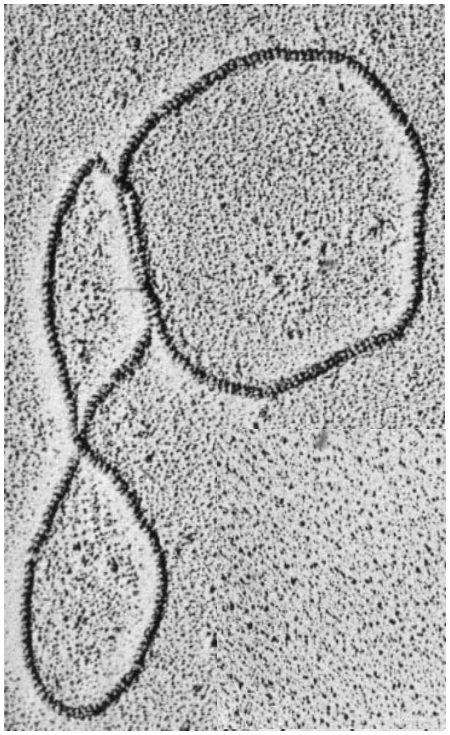
3 Topological Forms of DNA



supercoiled

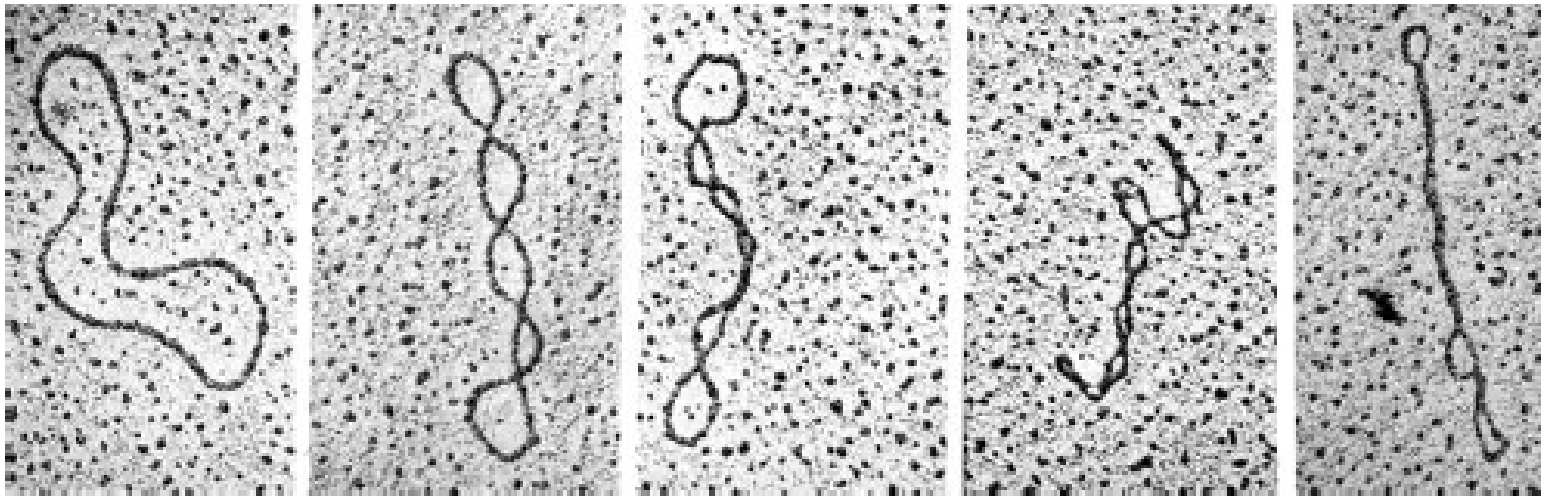


knotted



catenated

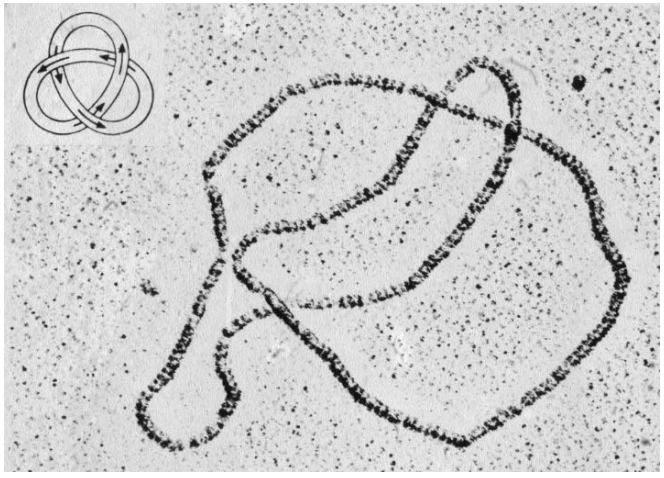
DNA Supercoiling



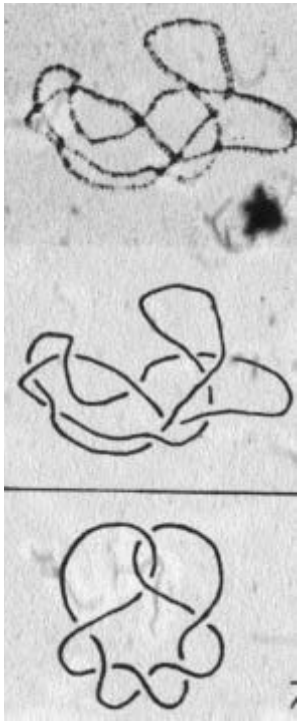
[Vinograd & Lebowitz, J Gen Physiol 49 (1966) 103-125;
Shore & Baldwin, J Mol Biol 170 (1983) 957-1008;
Horowitz & Wang, J Mol Biol 173 (1984) 75-91]

DNA Knot

Self-entanglement of a single DNA molecule

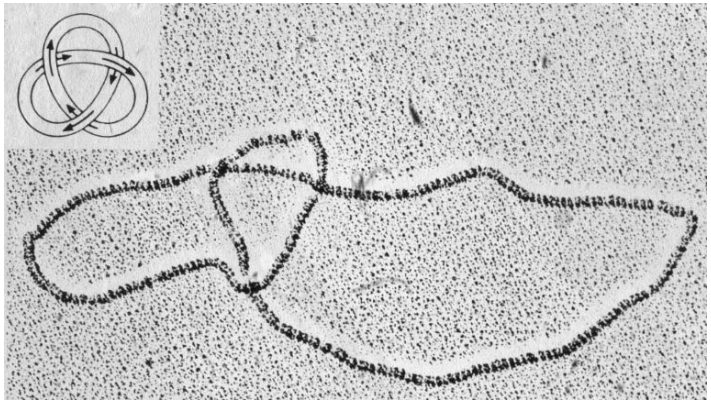


3_1^*



7_2

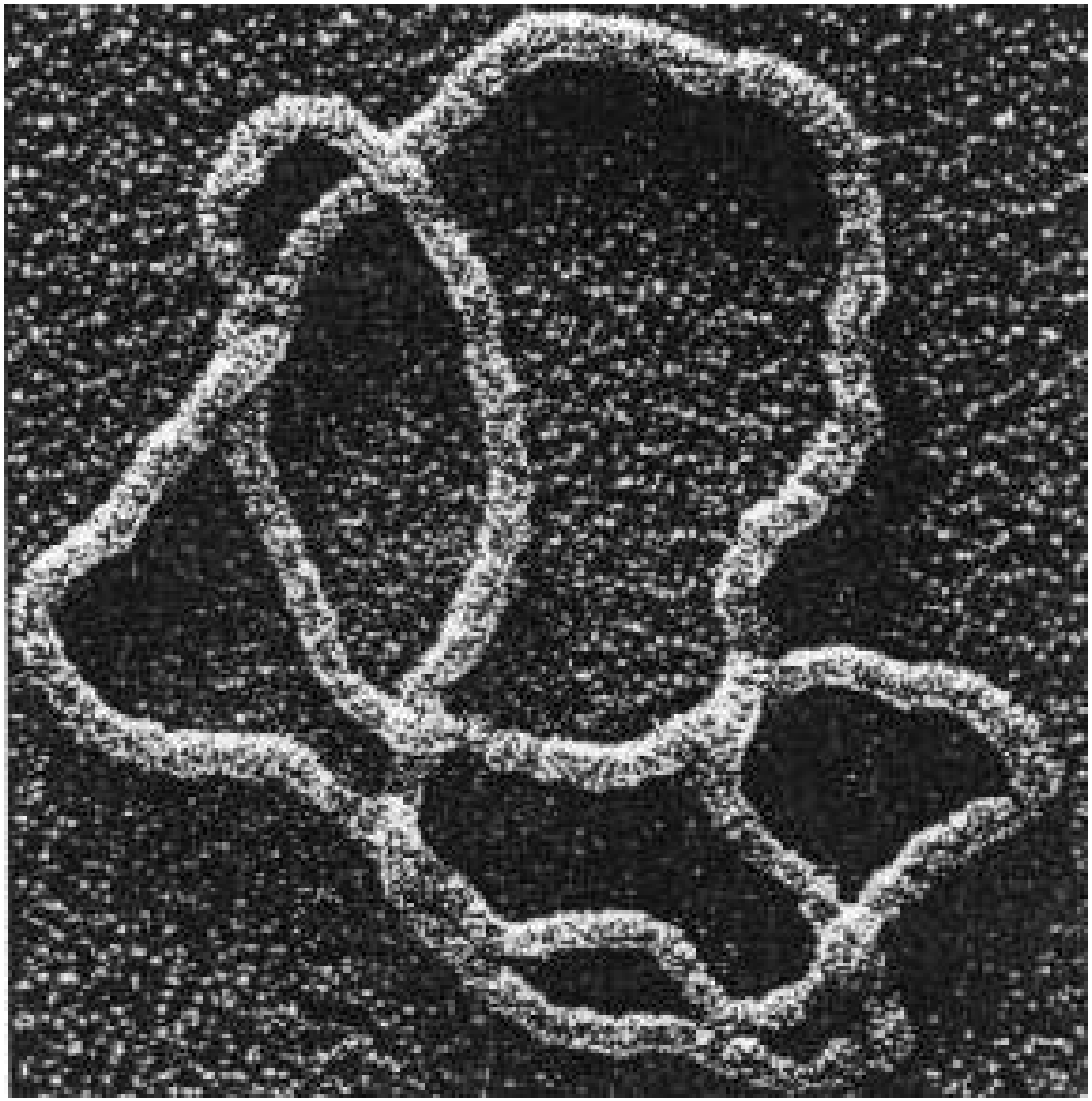
7_1^*



3_1

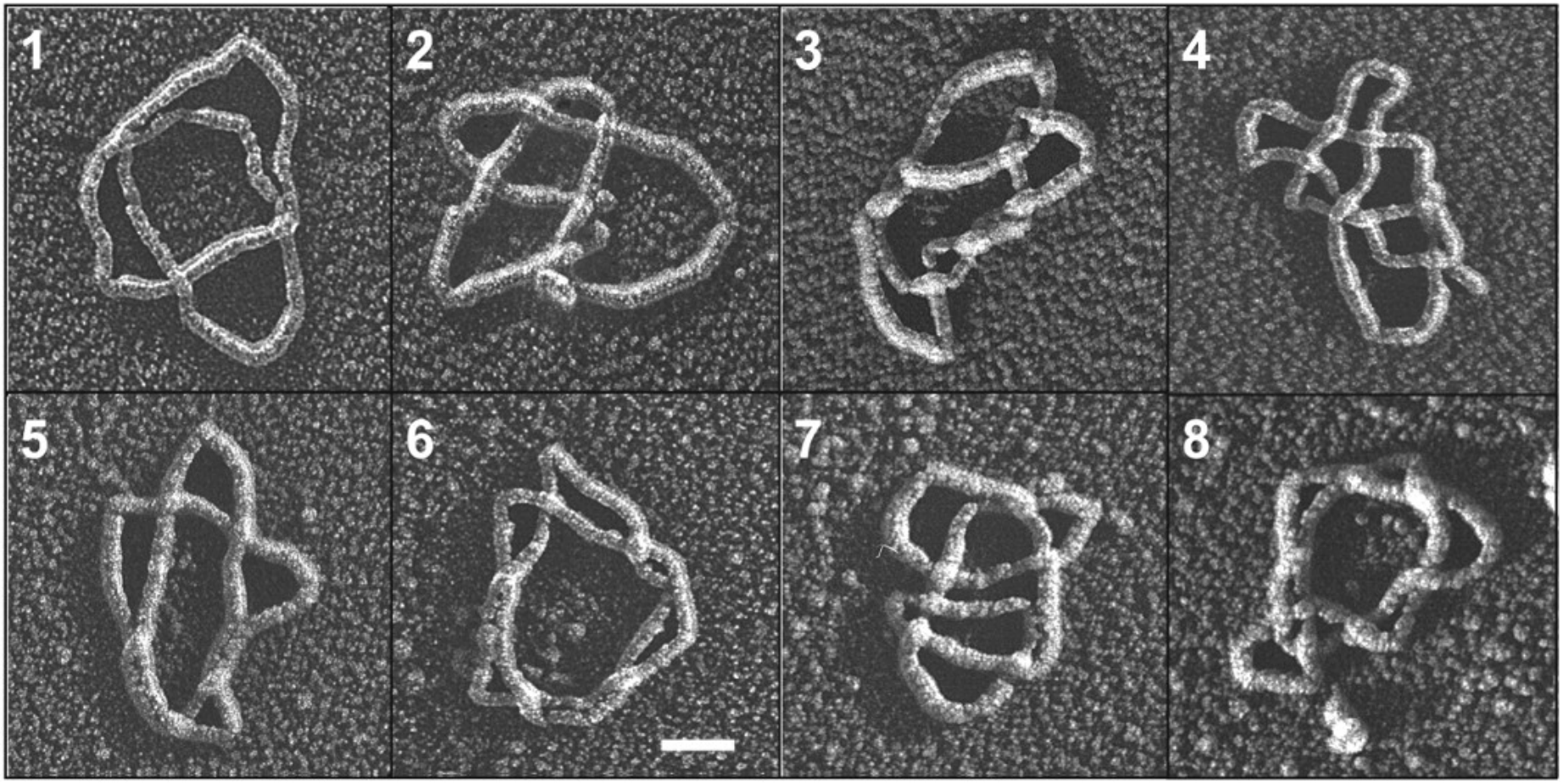
From http://lcvmwww.epfl.ch/~lcvm/dna_teaching_01_02/stasiak.html

DNA Knot



[From Sumners, D. (1995) *Notices of the AMS* **42,** 528.]

DNA Knots


(+) Trefoil
(-) 5 Twist
(-) 6 Twist
(+) 7 Twist
1
2
3
4
5
6
7
8
(+) 5 Torus
(+) 5 Torus
(+) Granny
(+) 7_3 Twist

DNA Knot

- the self-entanglement of a single DNA molecule

DNA Catenane

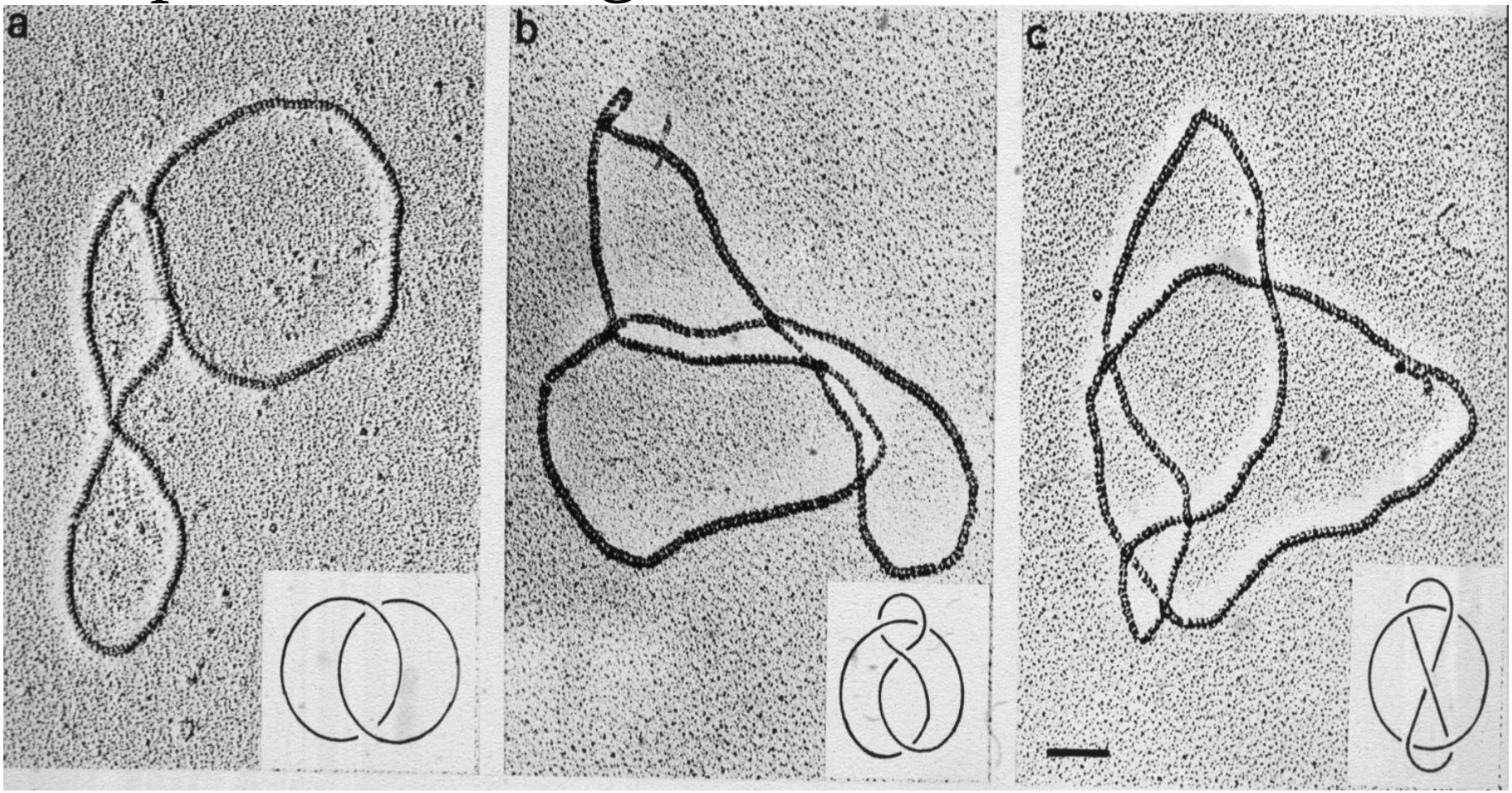
- two (or more) DNA molecules that are intertwined so that they can not be separated without breaking one of the strands

DNA Catenanes & Another Knot

Hopf link

Figure 8 Knot

Whitehead link



[From http://lcvmwww.epfl.ch/~lcvm/dna_teaching/stasiak.html.]

DNA Catenanes

[From http://lcvmwww.epfl.ch/~lcvm/dna_teaching/stasiak.html.]

Cellular DNA knotting is driven by:

DNA compaction

Topoisomerase reactions

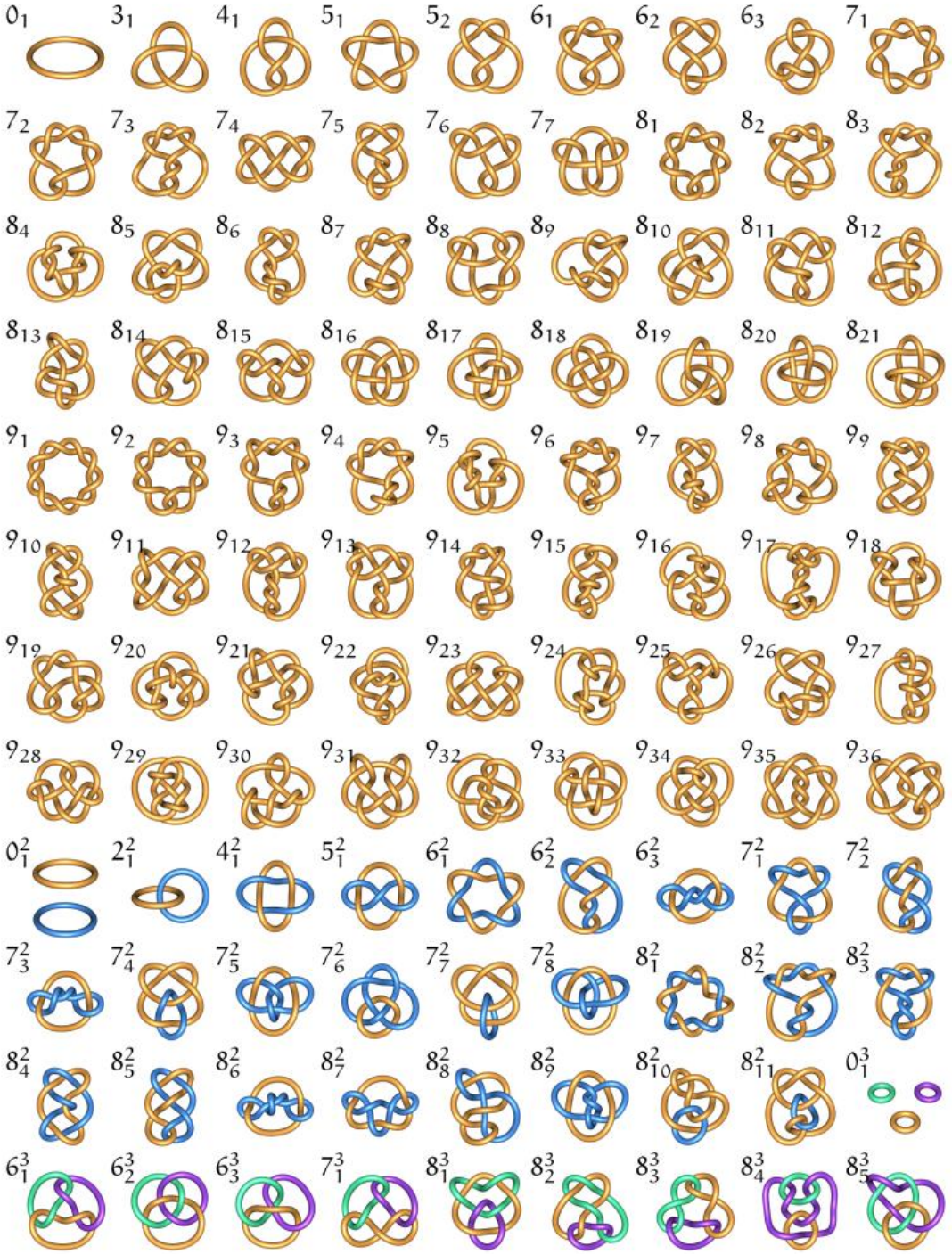
Replication

Supercoiling promoted strand collisions

Transposition, site-specific recombination, transcription

Rope Knotting Driven by You

Form one of these knots in your rope & tape the ends securely. Now add extra twists and trivial crossings to disguise the knot. Then pass it another group/person for them to try to identify



http://www.pims.math.ca/knotplot/zoo/

Consequences of DNA knotting:

1. Knots inhibit gene function by blocking replication and transcription

-In an essential gene, knots can be lethal

2. Knots induce DNA rearrangements

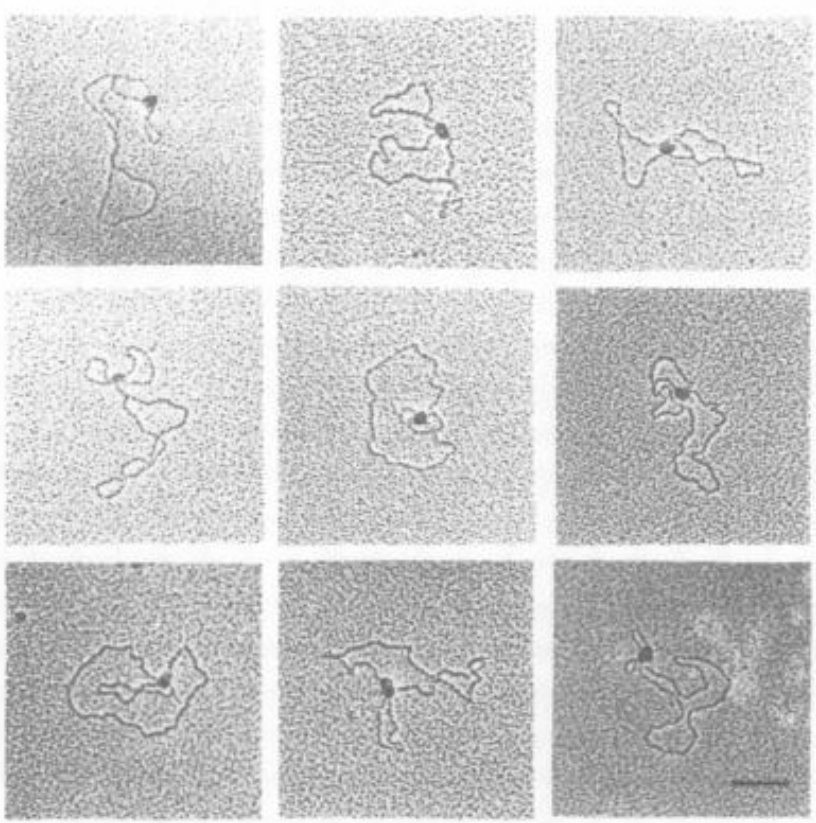
-Do knots promote evolution?

-Do knots account for the genomic instability associated with chemotherapy?

Topoisomerases

- Ubiquitous & essential
- Control DNA topology
- Pass one DNA strand through another via an enzyme-bridged transient break in the DNA
- Targets of antibiotics & anticancer drugs

Topoisomerase II Bound to DNA



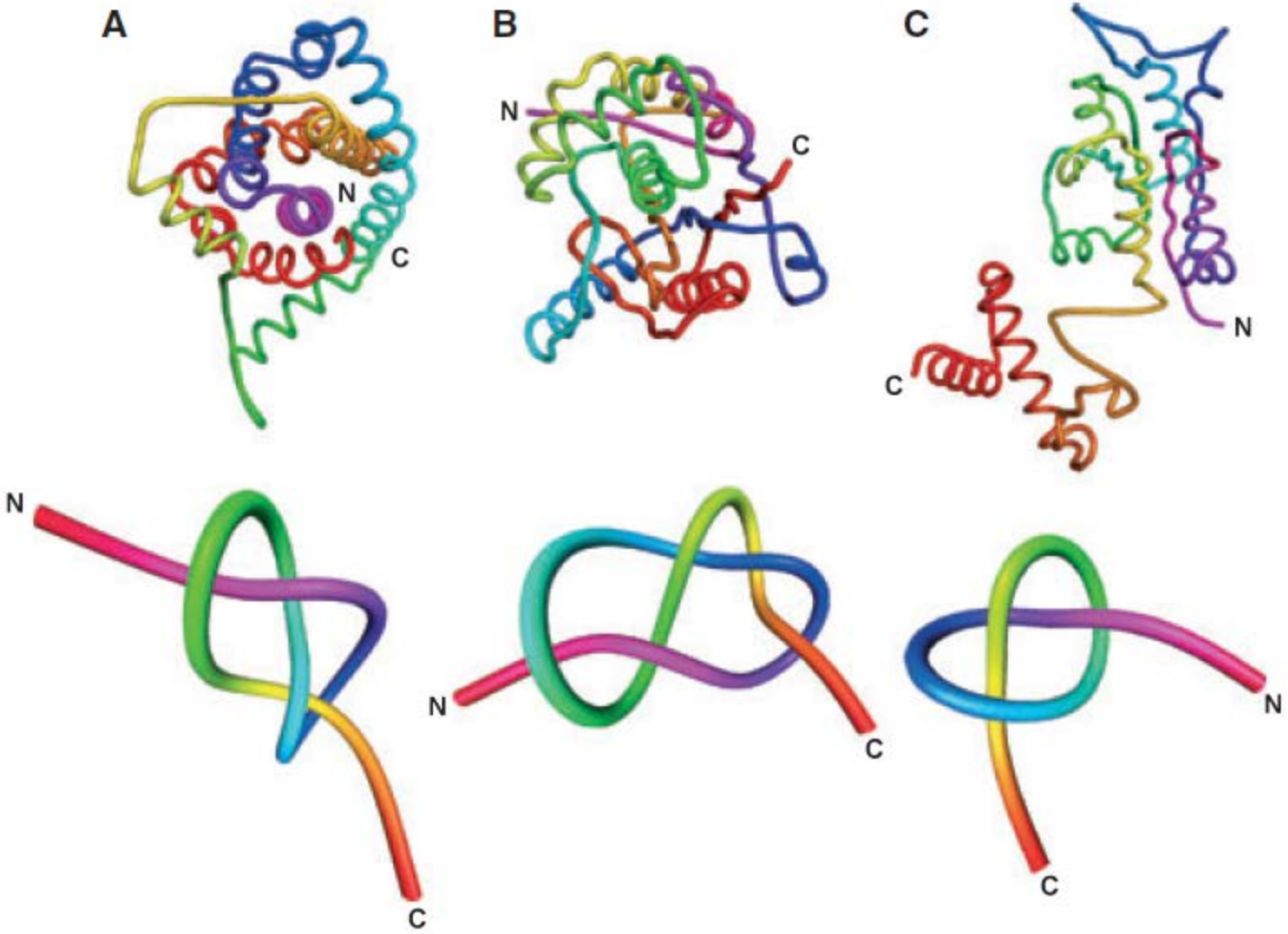
Eukaryotic topoisomerases recognize nucleic acid topology by preferentially interacting with DNA crossovers. Zechiedrich EL, Osheroff N. EMBO J. 1990 Dec;9(13):4555-62.

Knotenlöserin

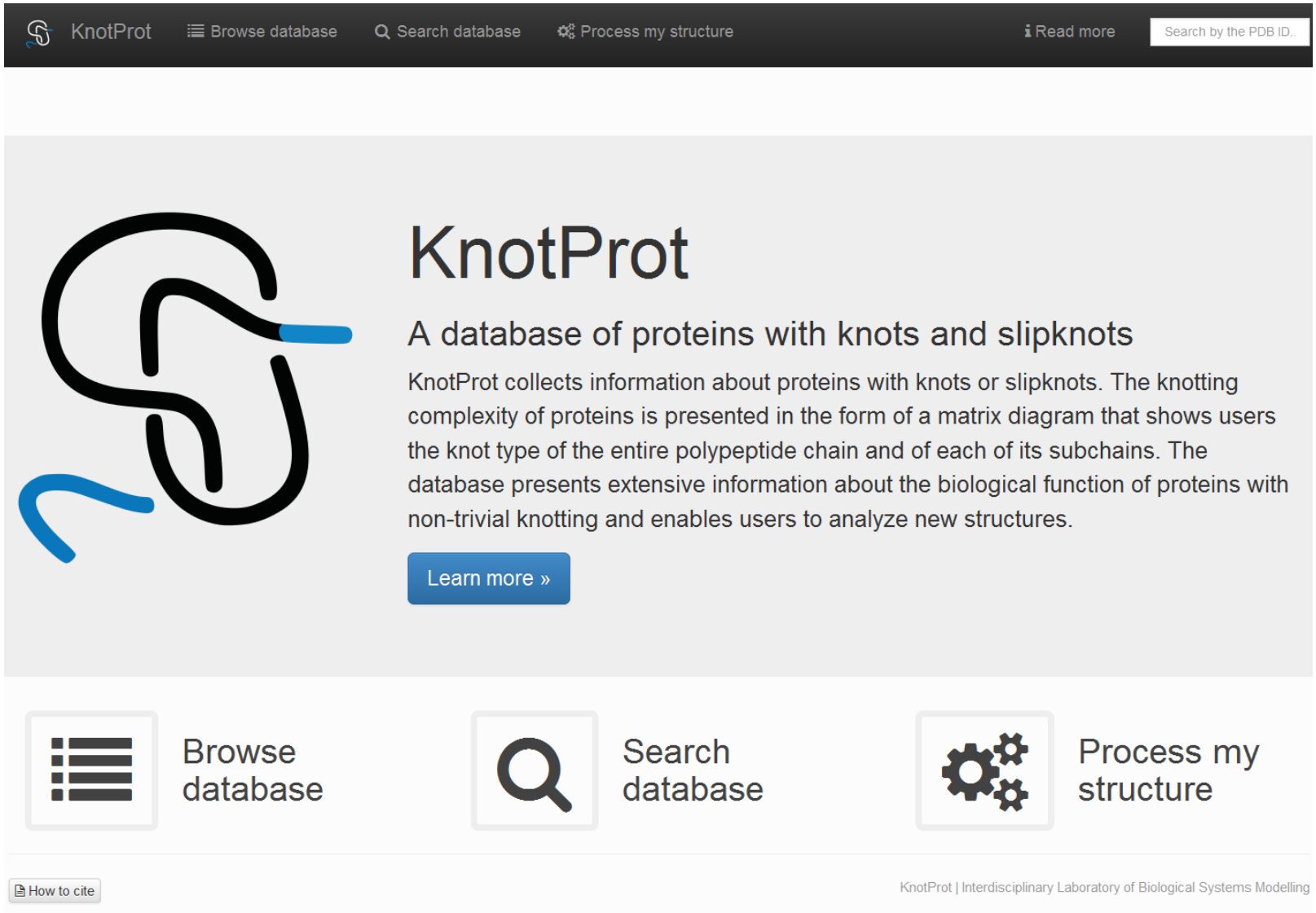
(a woman who loosens knots)
Artist: Irenaeus

http://campus.udayton.edu/mary//questions/yq/yq158.html

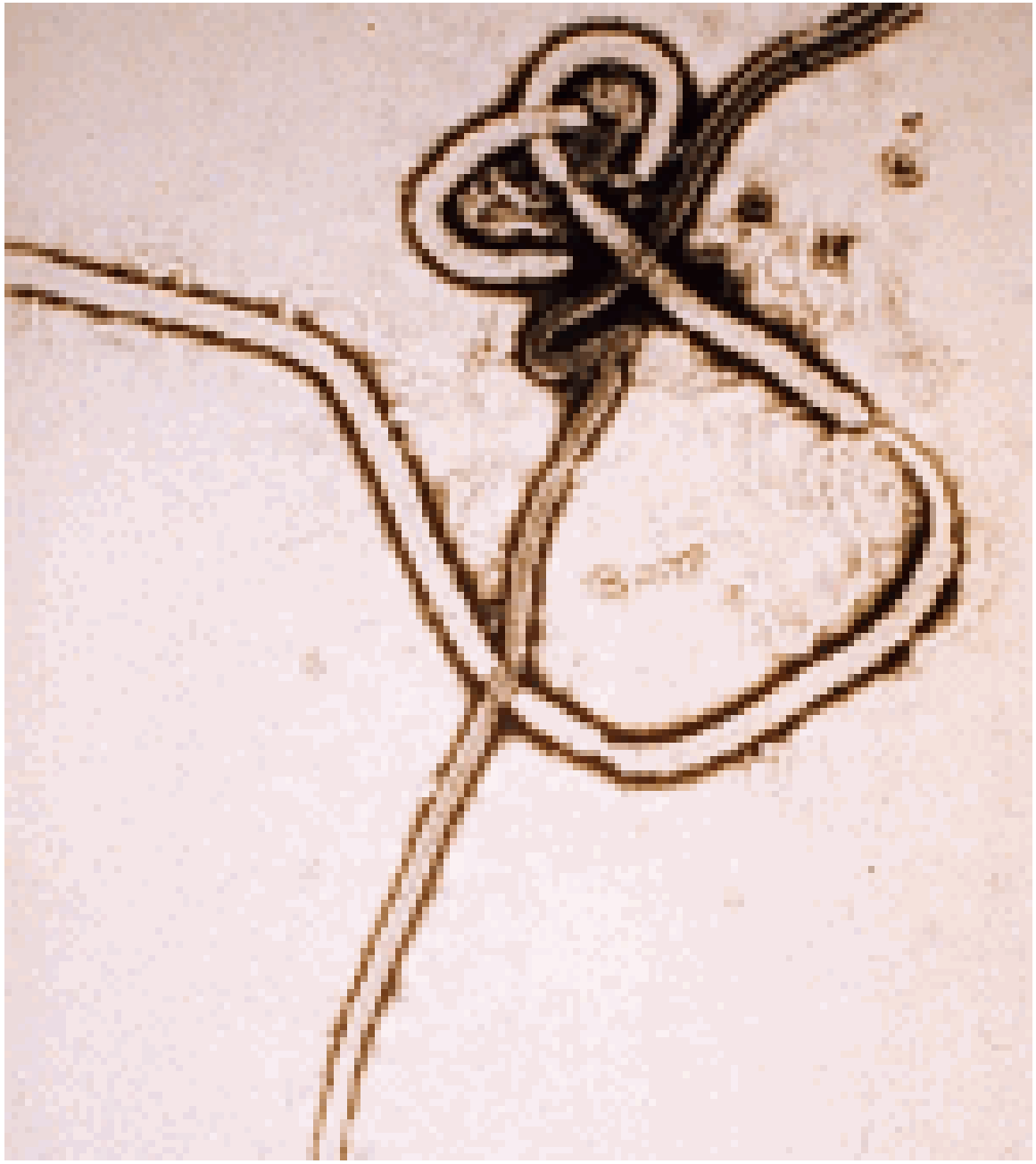
Knots exist in the native, folded state of some proteins



Mallam, A.I. FEBS J. 2009 Jan;276(2):365-75.



http://knotprot.cent.uw.edu.pl/



Ebola Virus

http://internetdev.state.sd.us/SDWebInfo/DOH/doh/LabBT/VHFviruses.htm

Thank you!

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