MTCircular

PIECE OF CAKE

Delectable Fractions and Decimals

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Math Teachers' Circle
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A Decade of Partnerships

Dear Math Teachers' Circle Network,

It's hard to believe that 2016 marks the 10th anniversary of the Math Teachers' Circle program! The very first Math Teachers' Circle, founded at the American Institute of Mathematics in 2006, was a collaboration among middle school mathematics teachers and mathematicians in the San Francisco Bay Area. They had the simple but innovative vision of a community where teachers and researchers could explore and enjoy mathematics together, forming lasting professional relationships in the process.

Ten years later, the Math Teachers' Circle Network has helped begin Math Teachers' Circles across the country. We continue to work closely with individual MTCs, as well as with emerging regional networks that provide vital infrastructure connecting nearby MTCs. Thanks to research supported by the National Science Foundation and others, we have learned a great deal about the program's benefits for teachers, which inclu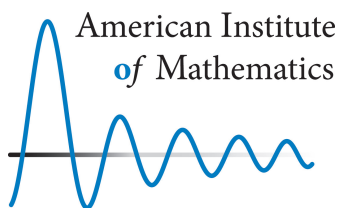de supporting productive mindsets about mathematics, Mathematical Knowledge for Teaching, professional engagement, and high-quality pedagogical practices that promote student learning. Many MTCs have developed creative ways of helping teachers make connections between their MTC and their classroom, for example as described by Gulden Karakok in "Good Beginnings." Hana Silverstein's article "Two-Way Street" describes some of the ways mathematicians benefit, too. In short, we have grown tremendously since those first meetings at AIM, and yet I am proud to say that we have also remained true to the original vision of building meaningful mathematical partnerships between teachers and mathematicians.

Between Heather Danforth's "Piece of Cake" classroom lesson on fractions and decimals and the "Flipping Pancakes" featured session by Katie Haymaker, we hope this issue will leave you hungry for more mathematics.

Happy problem solving!

Brianna Donaldson

Brianna Donaldson, Director of Special Projects

American Institute *of* Mathematics

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PIECE OF CAKE

Delectable Fractions and Decimals

by Heather Danforth
Helios School, Sunnyvale, California

There is never enough time to do everything I'd like to do with my students. I chose to develop this lesson because it accomplishes so many things at once: it gives students the opportunity to practice operations with fractions, to notice and explain patterns, and to review understandings of place value and number sense. It exposes students to sophisticated mathematical ideas, and it provides lots of practice with long division. What a rich opportunity!

I developed this lesson plan in July 2015 at the AIM Math Teachers' Circle Immersion Workshop for Teacher Leaders. Tom Davis and Mary Fay-Zenk (two of the co-founders of the AIM MTC) provided invaluable guidance during the workshop. Tom, especially, inspired this lesson and helped me write it. In addition to being deeply involved in Math Circles for teachers and students, Tom has written a number of materials for MTC session leaders – including the exploration of fractions and decimals on which this lesson is based. Tom's materials are available on his website, http://www.geometer.org.

Introduction to the Lesson Plan

In this lesson, students will explore the relationship between fractions and decimals. The lesson is appropriate for middle school students who are comfortable with pre-algebra, and it is extendable for advanced students. It provides an opportunity to work with many of the Common Core State Standards for Mathematical Practice, particularly MP1: Make sense of problems and persevere in solving them. No special materials are required, but calculators may be helpful for students who struggle with long division.

Objectives

Students will work with the Common Core State Standards for grades 7 and 8 in "The Number Sense" domain; in particular:

- 7.NS.A.2.D: Convert a rational number to a decimal using long division
- 8.NS.A.1: Convert a decimal expansion which repeats eventually into a rational number

Students will also employ the mathematical practices of exploring patterns and justifying their reasoning. In the process, they will have a research-like experience. Extensions will expose students to advanced concepts such as the sum of an infinite geometric series and fractions and decimals in bases other than 10.

The Cake Problem

"Imagine a giant cake here in our classroom. The first student is invited to eat 9/10 of the cake. The next student eats 9/10 of what is left of the cake. The next student eats 9/10 of what is left, and so on. If there are an infinite number of students in our class, each eating 9/10 of the remaining slice of the cake, will we ever completely eat the cake?"

Help students visualize the problem mathematically: the first student eats *9/10* of the cake, leaving *1/10* of the cake for the next student. The second student eats *9/10* of the remaining portion, or *9/10* of *1/10*, which equals *9/100*. How much of the cake has been eaten? (*9/10 + 90/100 = 99/100*). How much of the cake remains? (*1/100*).

After a few rounds of this, ask students if they agree that the cake will get eaten entirely. Designate one side of the classroom as "Agree," the other side as "Disagree," and invite students to express their opinion by moving to one side of the room or the other. If they are unsure, they can stand somewhere along this continuum.

Provide a short period for students to defend their positions, try to convince others, and move along the continuum if they desire.

"Let's come back to this question later, and begin with some easier examples."

Exploration #1:
Converting Fractions into Decimals

"To understand this problem better, we need to understand the relationship between fractions and decimals."

Write out the first six unit fractions on the board (*1/2, 1/3, 1/4, 1/5, 1/6, 1/7*). As a group, using long division, convert each fraction into its decimal expansion. Students will begin to see that some fractions terminate when expressed as a decimal (such as *1/2 = 0.5* and *1/4 = 0.25*), but other fractions repeat forever (*1/3 = 0.333…* and *1/7 = 0.142857142857…*). Some fractions have a non-repeating part followed by a part that repeats forever, for example, *1/6 = 0.1666….* What's going on here?

Here are some questions you can ask students:
- What do you notice about the decimals in this sequence?
- Why do some decimals terminate?
- Why do some fractions repeat?
- Can we turn all fractions into decimals?

Examples of student work from Danforth's use of this lesson plan in her own classroom. From top, students explore decimal expansions of sevenths, sequences of unit fractions, and converting the fraction 1/98 to a decimal.

Exploration #2:
Converting Decimals into Fractions

"We can convert fractions to decimals using long division, but how do we convert decimals to fractions?"

Let's go back to our first six unit fractions, represented now in decimal form: *0.5, 0.333…, 0.25, 0.2, 0.1666…, 0.142857….* You may choose to start with the terminating decimals in the sequence, in order to present a few easier examples.

What is the fraction representation of *0.5*? We know that it is *5/10 = 1/2.*

What is the fraction representation of *0.25*? We know that it is *25/100 = 5/20 = 1/4.*

What is the fraction representation of *0.2*? We know that it is *2/10 = 1/5.*

What is the fraction representation of *0.333…*? We know, because we've memorized it, that it is *1/3.* If we didn't know that it was *1/3,* how could we figure it out?

We might wish that *1/3* could be represented as a terminating decimal *0.3.* That's a pretty good approximation. However, it is an approximation and not an equality, because *0.3 = 3/10* but *3/10* does not equal *1/3.*

How can we make the approximation better? It is possible to find the difference between any decimal and a fraction by subtraction. For example:

1/3 - 0.3 = (10/30-9/30) = 1/30
1/3 - 0.33 = (100/300-99/300) = 1/300

The difference between *1/3* and *0.333* is *1/3000.* The difference between these representations can get very, very small, if we take lots of 3's. But if we take any terminating decimal, then no matter how many 3's there are, there will still be a positive difference when we subtract it from *1/3.* It will only be equal to *1/3* if the repeating digit continues infinitely. If it stops at some point, it will not be exactly equal to *1/3.* Why is that? (Hint: What happens if you multiply *0.333...* by 10? How is the answer related to the original problem of converting decimals to fractions?)

Write the following statement on the board: "0.999… is equal to 1." Do students agree or disagree with this statement? If 0.999… does not equal 1, what fraction is it equal to? Ask students to explain to one another and then to the group why this question is the same as the cake situation. This is an advanced mathematical concept!
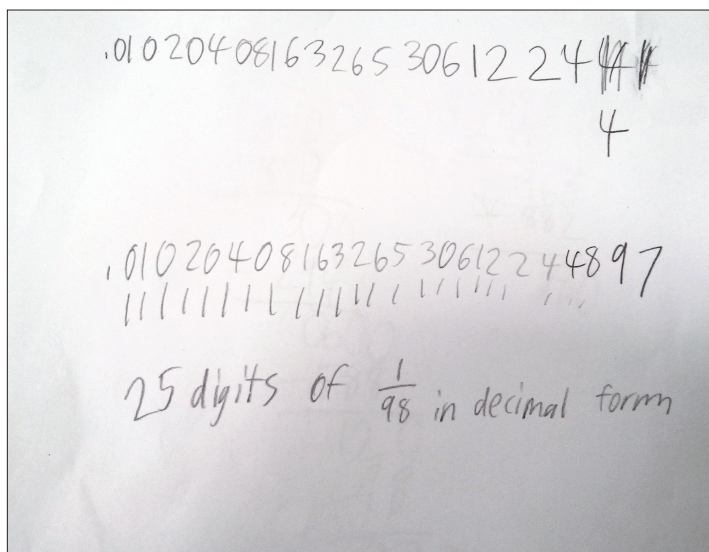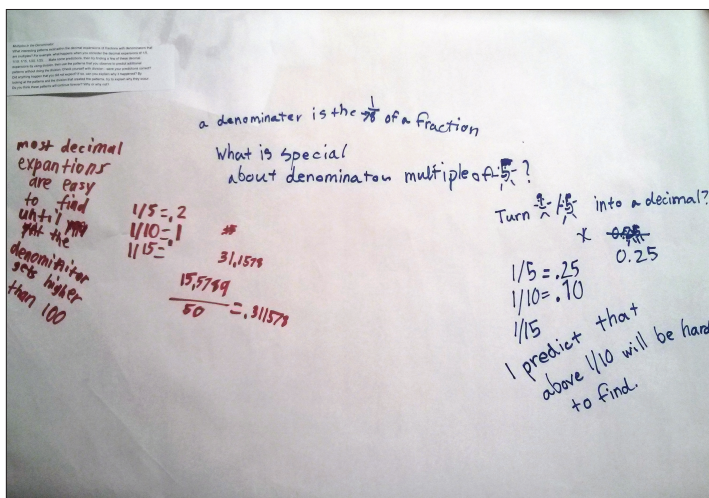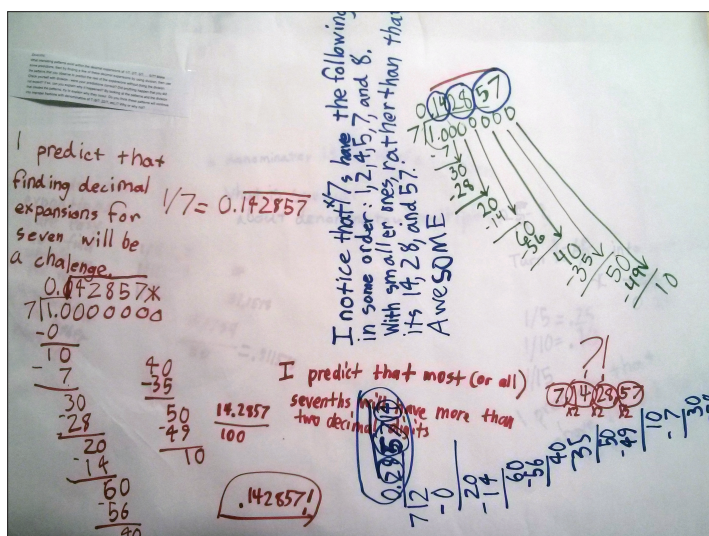
Work Time/Student Support

At this point, if time permits, give students about 30 minutes to work in pairs or small groups to investigate one or more of the research questions in the sidebar at right. Remind them to be organized and methodical. Suggest that they set up an organizational scheme, such as a table or T-chart, to manage their data and hunt for patterns. During this time, you may circulate and ask questions to encourage student thinking.

Debrief/Wrap-up

Bring the class together to debrief. Invite students to share the progress they made on their research questions. Highlight the problem-solving strategies they used (e.g. exploring patterns, justifying their reasoning) and the concepts they learned (e.g., the relationship between fractions and decimals). They may be interested to know that the process they engaged in is exactly what research mathematicians do!

Finish by revisiting the Cake Problem and the related question: *"Is 0.999… equal to 1?"* Give students a moment to check in with their neighbors about where they stand on the question.

If we look at the sequence of numbers *0.9, 0.99, 0.999, …*, the pattern we notice is that the first number in the series is *1/10* away from *1*; the second is *1/100* away from *1*; the third is *1/1000* away from *1*, and so on. The sequence gets arbitrarily close to *1*. So, we conclude that the number *0.999…*, with an infinite number of nines, is equal to *1*. Why? Because on a continuous number line, if *0.999…* were less than *1*, there would always have to be another number between *0.999…* and *1*. This is called the limit of a sequence. What does it mean for the limit of a sequence to equal *1* (or any number at all)? This is a philosophical question for students to continue to ponder!

Extensions

There are many opportunities for advanced students to further develop the concepts investigated in this lesson. Some of these include using the sum of an infinite geometric series to convert from a repeating decimal to a fraction; and investigating different number-base systems to see which fractions will terminate and which will repeat indefinitely. More extensions are available at **www.mathteacherscircle.org**.

Research Protocol

*Your students can get a taste of math research by repeating these two steps: **think** about an interesting unsolved problem, and **do something** to try solving it.*

1. **Think:** What do you think these fractions will look like as decimals? Write your prediction.
2. **Do something:** Use long division to turn the first few fractions into decimals.
3. **Think:** Write down any patterns you notice. Are they what you predicted? What do you think the rest of the decimals will look like?
4. **Do something:** Use long division to turn the rest of the fractions into decimals.
5. **Think:** Were your predictions correct? Did anything unexpected happen? Look at the patterns and try to explain why they occur. Do you think these patterns will continue forever?

Research Questions

Sevenths

What interesting patterns do you notice in the decimal expansions of *1/7, 2/7, 3/7, … 6/7*? Do you think these patterns will continue into improper fractions with denominators of 7 (*8/7, 22/7*, etc.)? Why or why not?

Negative Powers of Two

Negative powers of two (2^{-1}, 2^{-2}, 2^{-3}, …) create interesting decimal expansion patterns. What patterns do you notice in the decimal expansions of *1/2, 1/4, 1/8 1/16, 1/32, ...*?

Terminating and Repeating

Looking at the first ten or twenty unit fractions, what do you observe in those fractions that terminate in zeros as opposed to those that repeat? Why do you think that these fractions terminate in zeros? Do you expect that your observations will work for all unit fractions? Why or why not?

Good Beginnings
Supporting "Principles to Actions"
by Gulden Karakok

The Northern Colorado Math Teachers' Circle, established in 2011, aims to structure its monthly evening sessions and summer residential workshops around three core features of effective professional development, described by Desimone and her colleagues:

- Explicit focus on mathematics content through problem-solving activities;
- Active learning environments in which teachers engage in meaningful discourse to improve their learning and teaching practices; and
- Activities coherent to teachers' work (e.g., curriculum they teach, school/district goals).

All of our sessions are designed to improve teachers' problem-solving skills in the context of significant mathematical content in a collaborative learning environment. In addition, we have been engaging teachers in discussion on the ways in which they could incorporate problem-solving lessons in their classrooms.

Most recently, to align our MTC activities with teachers' classroom work, we started to emphasize and implement the eight *Mathematics Teaching Practices* that are outlined in the 2014 National Council of Teachers of Mathematics' "Principles to Actions" document. In brief, these practices are:

1. Establish mathematics goals to focus learning;
2. Implement tasks that promote reasoning and problem solving;
3. Use and connect mathematical representations;
4. Facilitate meaningful mathematical discourse;
5. Pose purposeful questions;
6. Build procedural fluency from conceptual understanding;
7. Support productive struggle in learning mathematics; and
8. Elicit and use evidence of student thinking.

> "THIS OVERLAP BETWEEN WHAT PARTICIPANTS EXPERIENCED AS LEARNERS AND WHAT THEY WANT TO IMPLEMENT AS TEACHERS IS AN AREA THAT WE HOPE TO EXPLORE MORE."

These eight teaching practices provide a research-based framework for teaching mathematics in every lesson. Furthermore, these practices guide teachers' actions to connect math content standards to the Standards for Mathematical Practice.

In 2015, we held our third summer workshop (in collaboration with Denver's Rocky Mountain MTC) and re-structured our sessions to foster an environment in which these practices are implemented and discussed explicitly. Similar to our last two residential workshops, participants worked on challenging math problems focusing on various content areas during the morning and afternoon sessions. In contrast to previous workshops, at the beginning of our first session, we introduced these eight teaching practices and had a very brief discussion of the framework. This introductory discussion was followed by our usual problem-solving activity. At the end of each day, we explicitly asked our participants to reflect on these teaching practices as learners. More precisely, we asked teachers to report on the teaching practices that they observed being implemented by

the facilitators and that impacted their learning as problem-solvers.

Each evening, similar to our previous summer workshops, participants met for at least two hours to engage in "Connections to Curriculum" sessions. At these sessions, participants discussed the ways in which they could implement problem solving in their classrooms. We asked them to modify tasks from our sessions or develop a new task to implement in their classrooms. To guide our participants' task development, we assigned them to read and discuss the *Task Design Framework* by Smith and Stein (1998). During the 2015 workshop, we also asked participants to develop action or lesson plans that would structure the delivery of the task using the eight teaching practices. Participants worked in groups of two or three to develop and design their tasks and lesson plans. These self-selected groups of teachers either had similar curricula or taught at the same grade level. Almost all groups selected an exercise or a specific math topic from their curriculum to design or develop a task with higher-level cognitive demand. In their action or lesson plans, teachers highlighted the need to emphasize all eight teaching practices.

Overall, our participants reported that they noticed or experienced all of the eight teaching practices being implemented by the facilitators during the daily problem-solving sessions. Interestingly, the third practice—"Use and connect mathematical representations"—was mentioned most frequently (43 percent of 115 reflections). We also noticed that while participants aimed at using all eight teaching practices in their action and lesson plans, groups again tended to explicitly mention implementation of the third practice most frequently.

This overlap between what participants experienced as learners and what they wanted to implement as teachers is an area that we hope to explore more. Even though, as facilitators, we aimed at implementing all eight teaching practices, our participants' reflections indicate that the implementation of these practices requires explicit discussions, especially if we want all of these teaching practices to play a role in mathematics classrooms. ◧

Summer 2015 immersion workshop for the Northern Colorado and Rocky Mountain MTCs. The author is pictured at far right, wearing a pink shirt.

Two-Way Street
Old Friendships, New Collaborations
by Hana Silverstein

Laura Janssen wants her seventh-graders to think deeply about simple things they normally take for granted. Her students are adding integers, and Janssen has them close their workbooks. She passes out colored plastic chips. *If these chips represent integers, what does it look like to add integers? Can you develop your own procedure for adding? When we return to the workbook, which method will you choose to solve the problems?*

Janssen is in her fourth year as a middle school teacher, and this is the first year she has been able to integrate problem solving into her classroom, thanks in part to her school district's decision to dedicate 90 minutes per day to math. Part of Janssen's confidence comes from her close collaboration with two mathematicians: her husband, Mike Janssen, and his colleague, Tom Clark, both of whom she met eight years ago at the University of Nebraska-Lincoln. Laura Janssen was an undergraduate math major. Her husband and Clark, both graduate students in mathematics, shared an office.

Tom Clark had entered graduate school after teaching for four years at a central California high school. As a teacher, he liked doing the upper-level math classes – statistics, pre-calculus, and AP classes – and he had the feeling that he'd prefer teaching at the college level. Plus, he adds, "After teaching for four years, I missed learning new things."

At the University of Nebraska, Janssen and Clark both got involved in the Lincoln Math Teachers' Circle. "I enjoyed problem solving and doing things with teachers," says Clark. "That's something I never had when I was teaching. No MTC, no professional development geared to math teachers. I decided I wanted to start an MTC wherever I ended up."

Meanwhile, Laura Janssen graduated with a bachelor's degree, and went on to get a master's and a teaching credential with the support of a Noyce Fellowship at the University of Nebraska. She started out teaching at a middle school in a low-income area of Lincoln. "It was super-overwhelming," she says. "I had five periods of kids and approximately 50-minute class periods." Many of the students were eating all three meals at school every day. Math wasn't a priority.

The school, however, had a strong extracurricular program: a Boys and Girls Club that worked with teachers to provide after-school programming and homework help. With the support of the Boys and Girls Club, and with a generous supply of snacks and juice, Janssen thought she might be able to interest students in an after-school math club.

She told Tom Clark about the idea. With permission from his boss, Jim Lewis (the Principal Investigator of the Nebraska Noyce grant), Clark decided to help out.

"The math club was very successful," says Janssen. "It was where I was able to start implementing things I learned in my master's degree program. I could do interesting problems, like finding the border you get when you line n hexagons up next to each other."

"There were some incredible moments," says Clark. He remembers a session where students calculated the number of tiles around a rectangular pool; he was blown away when the seventh graders started figuring out different ways of representing numbers.

"It was such a good experience for my students to be around a mathematician, and see him as a person who

was funny and made addition errors, too," Janssen says. "And for me, it was great practice to start doing some of those problems in a math club, because now I have time to do them in my classroom."

Now, Janssen teaches in South Sioux City, a town on the Iowa border on the northeastern edge of Nebraska. About 50 miles away, her husband teaches at Dordt College in Sioux Center, Iowa. (The stipulations of Janssen's Noyce Fellowship require that she continue teaching in Nebraska.)

This middle school, like her first one, serves a large number of first- and second-generation Latino immigrants. Now, however, Janssen has the luxury of 90-minute class periods. "I have time to do some deep conceptual things, and then build on to the procedural, all in the same day," she says. (For instance, using integer chips to explore what addition looks like!)

Janssen frequently consults her husband when planning her classes. "There are so many times when I go to Mike with a question about mathematics, like, 'I want to teach something conceptual, is this a good way to motivate it?' And it's a two-way street. Things that are good for seventh graders instructionally are also good for college students."

Serendipitously, Tom Clark also ended up at Dordt College, where he teaches a mix of engineering students, math majors, and math education majors. "I like having pre-service teachers in my classes," Clark says. "I feel like I can tell them, 'Hey, we're learning this cool math, and here's where you're going to use it as teachers.'"

As promised, Clark started the Northwest Iowa Math Teachers' Circle in late 2014. He likes serving a rural area, but it can be a challenge. "This is a town of 8,000; we maybe have four math teachers in the city," he says. Many teachers drive around half an hour to attend MTC meetings, and attendance fluctuates from month to month. But Clark is committed to the Circle;

Stock Photo: Travis D. Kosai, Stock.Xchng. Session photos courtesy of Tom Clark.

Tom Clark takes the wheels in a session on "Bicycle Tracks" led by James Tanton at a recent meeting of the Northwest Iowa Math Teachers' Circle.

he feels like he has something to offer the teachers, in terms of deepening their mathematical understanding of the content they teach.

"I'll have teachers e-mail me and say, 'A kid came in with this calculus question; I can't figure it out.' I'll figure it out, type it up, e-mail it back to them. If they're the calculus teacher in their school, it probably means they're the only teacher at their school that knows calculus. Where else do they go?"

Clark also sees personal benefits to being part of a community of teachers. "I'm teaching college freshmen who, six months ago, were high school seniors," he says. His connections with local teachers help him stay current on what high school students are learning and struggling with these days.

Janssen attends most of the meetings, and helps facilitate discussions among the different tables. She says the MTC helps teachers see math with a broader perspective, and provides support for teachers who want to take this more expansive view of math into their classrooms despite pressure to focus on large-scale mathematics assessments. "If all you assess are straightforward procedural problems, then all you will teach are straightforward procedural problems," she says. "And if you're the only teacher in your building that's trying to do differently, you get questioned."

In the future, Clark hopes to see more teachers not just attending, but also presenting MTC sessions. "Laura has presented about graph theory in the game of SET," he says. "I think some of the other teachers who come say, 'Okay, here's a middle school teacher, and she's doing math.' I think that's encouraging; 'I can do math, too; I can take ownership of my discipline.'"

Clark says it's a cumulative effect: teachers get together once a month, learn new skills and problems, and eventually start presenting. They become experts; they grow professionally. They grow in their ability to reason, to explain, to be curious. "Curiosity is a really good thing to help students grow in, and help teachers grow in as well," says Clark.

"As a math teacher, I feel like I have such an advantage having this community of mathematicians," says Janssen. "I think it would be nice to give more teachers that advantage." ⊑

Mike Janssen and Laura Janssen, left, look on as session participants investigate the mystery of the bicycle tracks. Which way did the thief go? Find resources for this session at http://www.mathteacherscircle.org/newsletter.

Dispatches from the Circles
Local Updates from Across the Country

California •

The new **North San Luis Obispo County MTC** held its third meeting in October 2015 with teachers from Atascadero and Cayucos. The group put their heads together to tackle the Cookie Jar problem, adapted from *The Inquisitive Problem Solver* by P. Vaderlind, R. Guy and L. Larson: "*Fifteen cookie jars are numbered consecutively from 1 to 15. The number of cookies in each jar is equal to the number on the jar. A 'move' consists of choosing one or more jars and removing one or more cookies from each selected jar — but the same number of cookies from each jar. What is the minimum number of moves necessary to empty all the jars?*" After the group made some calculations and conjectures, Dr. Richard Grassl led an analysis of the results. The extension of the problem to *n* jars produced animated discussion. Co-leaders Bruce and Kathy Yoshiwara and current participants plan to expand the Circle to include teachers from Paso Robles and other nearby school districts.

– Contributed by Kathy Yoshiwara

Mississippi •

The **Mississippi Delta MTC** held its inaugural summer immersion workshop in July 2015, with funding from Domino's Pizza and Delta Dairy. Two teacher members, Whitney Janous and Penni Morgan, received 7-8 Math Supplemental Endorsements from the Mississippi Department of Education for three years of participation in the Delta Math Science Partnership program. Co-organizer Liza Cope received a $4,000 grant from the Phil Hardin Foundation to fund the annual meeting of Mississippi Association of Mathematics Teacher Educators at Delta State University. Cope also presented at the NCTM Regional Conference in Nashville, Tenn.

– Contributed by Liza Cope

New York •

Kovan Pillai has taken over as executive director of the **New York Math Circle (NYMC)**. Former director Japheth Wood stepped down in June 2015 to welcome new son Lihuel. Wood is back at Bard College as a visiting associate professor of mathematics, and uses his free time to develop the **Bard Math Circle**. NYMC member Fred Galli was awarded a 2015 Sloan Award for Excellence in Teaching Science and Mathematics, which "celebrates extraordinary teachers in New York City Public High Schools" (**http://www.sloan.org/fileadmin/media/files/civic/2015_Sloan_Teachers.pdf**).

– Contributed by Japheth Wood

Oklahoma •

Tulsa MTC leader Marilyn Howard and **Tulsa Girls' Math Circle** leader Donna Farrior received a grant of $1,500 from the National Association of Math Circles (NAMC). Farrior and Howard attended the NAMC's Mentorship and Partnership workshop at the University of Denver in September 2015, during which they helped mentor a Julia Robinson Math Festival (JRMF) for 300 students. They now plan to host their own JRMF in April 2016 at the University of Tulsa.

– Contributed by Marilyn Howard

Virginia •

Lockheed Martin donated $5,000 to the **Metro Atlanta MTC** to fund its 2015 summer immersion workshop. During the two-day workshop, held at Kennesaw State University, teachers worked on problems focused on flight, and took a tour of the Lockheed plant.

– Contributed by Virginia Watson

Alibegovic, Taylor to Present at This Year's MidSchoolMath National Conference

Alibegovic

Taylor

Emina Alibegovic (University of Utah and Utah MTC) and James Taylor (MTC of Santa Fe) are among the presenters at this year's MidSchoolMath National Conference in Santa Fe, N.M., a two-day conference focused entirely on middle school mathematics. According to the conference website, Taylor "has been working with math circles for students and teachers since 2005, and has had an interest in provocative and subversive mathematics education much longer" (**http://msm2016.sched.org**). Taylor will lead a 75-minute Teachers' Circle session along with a few of his MTC co-organizers. Alibegovic, who, according to the conference website, is "especially interested in teacher knowledge and geometry," will present a 45-minute session titled, "A Complete Circle of Cognitively Based Assessment."

Grant from North Carolina GlaxoSmithKline Foundation will benefit North Carolina teachers

The Smoky Mountain MTC leadership team: from left to right: Renee Stillwell, Sloan Despeaux, Michelle Massingale, Brooke Stillman, and Nathan Borchelt.

The North Carolina GlaxoSmithKline Foundation recently awarded a $200,000 grant to the Western Carolina University (WCU) Foundation, to support the Smoky Mountain MTC (Cullowhee, N.C.). WCU professors Sloan Despeaux and Nathan Borchelt, together with local teachers Renee Stillwell, Michelle Massingale, and Brooke Stillman, organized the Smoky Mountain MTC in 2014, inviting middle school math teachers from five counties across western North Carolina to "join with WCU professors for an ongoing dialogue about math," according to the university press release. "It's great to see these counties in the western area of the state working with this important math education program," said North Carolina GlaxoSmithKline Foundation board member Ran Coble. The grant will provide funding to sustain the Smoky Mountain group for the next five years and to create a network of MTCs across the state. The full press release is available at **http://news-prod.wcu.edu/2015/10/wcu-foundation-receives-north-carolina-glaxosmithkline-foundation-grant-to-support-math-teachers-across-region-state/**.

Documentary on Navajo Math Circles to Air on Public Television and at Joint Math Meetings

Navajo Math Circles by Zala Films is a new documentary about "the meeting of two worlds: that of some of the country's most accomplished mathematicians and math educators, with the children and teachers in the underserved, largely rural Navajo educational system" (**http://www.navajomathcirclesfilm.com**). Several of the directors of the Navajo Math Circles make an appearance in the film, including Bob Klein (Ohio University and SouthEast Ohio MTC), a leader of the Navajo Nation Math Camp in July 2014 and an expert on education in impoverished areas, and Tatiana Shubin (San Jose State University, AIM MTC), who introduced Math Circles to schools in the Navajo Nation in 2012. She has organized numerous mathematical events and math camps for Navajo children and teachers ever since. According to the film's website, "Both the Navajo leaders and the mathematicians from outside are searching for ways to link the math they are communicating to students with the traditional Navajo cosmological, architectural, and artistic concepts rooted in mathematics." *Navajo Math Circles* will appear on public television in 2016, and will be screened at the 2016 Joint Mathematics Meetings in Seattle on Wednesday, January 6, 2016, at 6:30 p.m, in Ballroom 6BC of the Washington State Convention Center. ⬚

Official film promotion photos of MTC leaders Klein, left, and Shubin, right. Photos by George Csicsery, courtesy of Zala Films. Used with permission.

Flipping Pancakes
by Katie Haymaker

The Pancake Problem is a sorting problem with connections to computer science and DNA rearrangements, which leads to discussions of algorithms, sequences, and the usefulness of approximations and bounds. The original problem was first posed by mathematician Jacob Goodman under the pen name "Harry Dweighter" (read it quickly) in 1975, and it has delighted mathematics enthusiasts (including an undergraduate Bill Gates) ever since! I first learned of the problem during a talk by the mathematics writer Ivars Peterson, and recently facilitated a session on it at the Philadelphia Area Math Teachers' Circle.

For this session, it is useful for each table to have at least four small disks of different sizes to simulate pancakes.

The Pancake Problem
The chef at the Philadelphia Area Mighty Tasty Cafe holds the record for making pancakes quickly. However, in her rush to get the pancake orders out to customers, she ignores how the stacks look as they leave the griddle. No two pancakes are the same size, and the chef tosses the pancakes directly from the griddle onto the plate. The waiter delivering the pancakes tries to rearrange the stacks on his way out of the kitchen, but since he is holding the plate in one hand and the spatula in the other, he is only able to

make one type of move in his quest to adjust the stack: he can stick the spatula somewhere in the stack and flip, in one motion, everything that sits above the spatula. Figure 1 shows an example of a legitimate move. The waiter's goal is
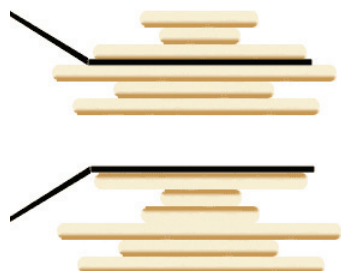
Figure 1: The pancake sorting operation. Image by Derrick Coetzee via Wikipedia.

for the stack of pancakes to increase in size from the top of the stack to the bottom, so by the time it gets to the customer it looks like a pyramid of pancakes. He notices that it is easier to do this for some stacks than for others. For example, occasionally the chef happens to hand him a plate where the pancakes are already in order, so he doesn't need to make any flips. Other times it takes him several flips to get the pancakes in order.

The waiter wants to make sure he can deliver the pancakes before they get cold, so he would like to know the worst-case-scenario number of flips he'd have to perform on any given number of pancakes. We call this number the "Pancake Number," and denote it by P_n, where n is the number of pancakes in the stack. At the Philadelphia Area Math Teachers' Circle, we started with a group discussion investigating P_n for small n. For example, if a stack has only one pancake ($n = 1$), zero flips are needed, so $P_1 = 0$. If $n = 2$, then the worst-case number of flips is one, to rearrange the stack with the larger pancake on top, so $P_2 = 1$. It turns out finding P_n is a hard problem as n increases! The Pancake Number P_n is currently unknown for more than 19 pancakes.

Tackling Small Cases
At this point, we broke into small groups to work on calculating P_3 and P_4. One of the first ideas that groups discussed was how to succinctly represent pancake sizes within a stack. A few groups started with "S, M, L" (small, medium, large) for the stack of three, and then switched to a number representation (1, 2, 3, 4) once they started to work on four pancakes. As groups moved forward with the problem, they found they needed to discuss and demonstrate whether they were using an efficient flipping strategy for each new stack. Groups also discovered that, although there is a unique stack that requires the worst-case number of flips for

n = 3, for each value of *n* larger than 3 there are several different stack configurations that require the worst-case number of flips. For example, for *n = 4*, there are three different configurations of pancakes that would require four flips to rearrange. What are they?

Further Leaps and *Bounds*

At this point, we challenged groups to come up with a general set of steps—an algorithm—for rearranging that would work on any stack of pancakes. We also asked how many steps this algorithm would take on *n* pancakes in the worst case.

One such algorithm is as follows: Find the largest pancake in the stack. Place the spatula under the largest pancake and flip so that it is now on the top of the stack. Next, place the spatula under the whole stack and flip so that the largest pancake is on the bottom. The second stage of this process is to find the second-largest pancake and place the spatula under it. Flip so that it is now on the top. Place the spatula just above the largest pancake (below the top *n-1* pancakes), and flip so that the second-largest pancake is now in the correct position. Repeat until all the pancakes are in the correct position. The worst-case number of flips for *n* pancakes using this algorithm is *2(n-2)+1=2n-3*. Each pancake takes two flips to get in the correct position, except for the final two pancakes, which only require one flip.

This algorithm provides us with a ceiling, or an upper bound, for P_n for each *n*. This is useful because, even though we may not know P_n exactly, we know that the waiter would be able to rearrange any stack of *n* pancakes in at most *2n-3* flips. The waiter would be happy to hear this news, because it means even a large stack of pancakes can be delivered before his shift ends. Whether they would still be warm is unknown!

During our session, groups came up with the same upper bound through a recursive argument. Their logic: It takes two flips to get the largest pancake on the bottom of the stack, and after that you are dealing with only the upper stack of *n-1* pancakes. Therefore, P_n is at most $P_{n-1}+2$, when dealing with more than three pancakes.

Philadelphia Area MTC participants ponder pancakes.

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Continued from page 17.

A final question we discussed as a whole group was how to obtain a floor, or a lower bound, on P_n. The challenge is to come up with a configuration of pancakes that you know would take at least n flips, no matter which flipping strategy is used. Participants had ideas about the "worst" stacks based on their work with four pancakes, so we tried to come up with a worst-case stack for five pancakes, too. One idea was to have the stack as "deranged" as possible. For example, if the correctly ordered smallest-to-largest stack is represented by (1, 2, 3, 4), then the stack (2, 4, 1, 3) has none of the adjacencies of the final desired stack, including the plate being next to the largest pancake as one adjacency. It takes at least four flips to rearrange this stack. In general, it takes at least n flips to rearrange such a stack, because it takes at least one flip to fix each adjacency. This provides a lower bound for P_n.

Burnt Pancakes and Beyond

How many stacks of size n require a certain number of flips? For example, how many stacks of four pancakes require zero flips, one flip, two flips, etc.? There are many additional questions to investigate, including the "Burnt Pancakes" Problem Circle puzzle below.

Katie Haymaker is Assistant Professor of Mathematics at Villanova University. This session was prepared with the cooperation of the Philadelphia Area Math Teachers' Circle Leadership Team: Catherine Anderson, Steve Bartholomew, Kathy Boyle, Aimee Johnson, Amy Myers, Josh Sabloff, and Josh Taton.

Problem Circle: Burnt Pancakes

If the pancakes are burnt on one side, the stack that gets sent out to the customer should not only be in order, but it should also have all the burnt sides facing down. What is the worst-case number of flips for two, three, and four burnt pancakes?
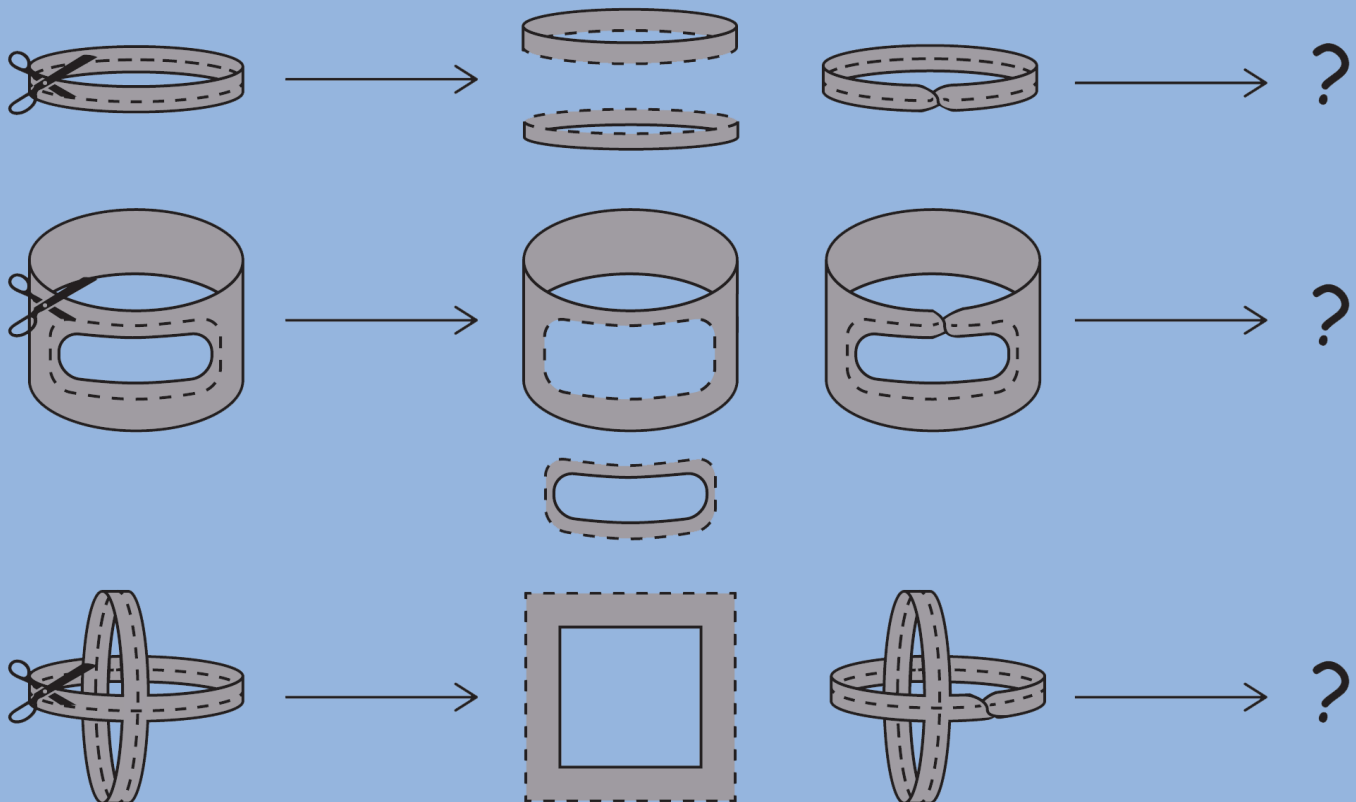
Pancake Photo: Alicia Solario, Stock.Xchng.
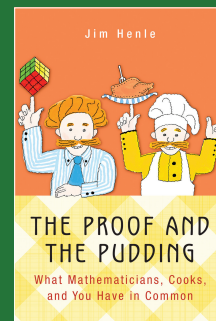
Math without Words

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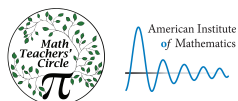
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